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By

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From

Inzhernoi Sbornik, 16, 119-48 (1953)
Academy of Sciences, SSSR, Institute of Mechanics

Redstone Scientific Information Center

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Translated from the Russian by
Ingeborg V. Baker

Translation Branch
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Directorate of Research and Development
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Redstone Arsenal, Alabama

TRANSLATOR'S NOTES

Whenever the Russian symbols кг are encountered in the text and expressions, these should be read as kg i.e. kilograms

When the Russian $_{кр}$ is found this should be read as $_{cr}$ i.e. abbreviation for CRITICAL.

The small Russian ш denotes RIB.

Russian alphabetical letter Б is equivalent to Latin B.

Г is equivalent to Latin G.

CONCERNING THE STRENGTH AND STABILITY OF CYLINDRICAL BIMETALLIC SHELLS

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E.I. Grigolyuck

The strength and stability of an axially symmetric, cylindrical, bimetallic shell is investigated in this article. The shell is assumed to be thin and elastic. It is considered that the hypothesis of incompressibility of the layers and of the non-deformability of the normal element retain this force also for the bimetallic shell /1-3/. Separately investigated was a shell of an infinite and finite length, and the so-called condition of reduction when applicable to the shell. Examples of calculation of shells down to the determinants of displacements and stresses, are given. In a particular case, the results for a homogeneous shell are obtained. Axisymmetric stability of the shell is investigated by linear arrangement.

1. Basic Equations of the Problem. Let the radius of the surface seam of the cylindrical, bimetallic, thin-walled shell equal R; its length, l; the thickness of the internal and external layer correspond to δ_1 and δ_2; x be the interval from the left edge of the cylinder to the cross section; z, the positive distance from the surface of the seam along the thickness of the shell if directed toward the center of the curvature. The element of the shell is shown in Fig. 1. During axisymmetric deformations in the cross sections of the element, normal σ_1 and tangent τ_{13} stresses occur. In the meridianal sections only the normal stress σ_2 occurs.

Internal forces relative to unit length of the seam surface: the bending moments M_1 and M_2, the normal forces N_1 and N_2 and the cross-sectional force Q_1, are shown in Fig. 1. They are associated with stresses by formulas

$$M_1 = \int_0^{\delta_1} \sigma_1^{(1)} \left(1 - \frac{z}{R}\right) z\,dz + \int_{-\delta_2}^{0} \sigma_1^{(2)} \left(1 - \frac{z}{R}\right) z\,dz \approx \int_0^{\delta_1} \sigma_1^{(1)} z\,dz + \int_{-\delta_2}^{0} \sigma_1^{(2)} z\,dz$$

$$M_2 = \int_0^{\delta_1} \sigma_2^{(1)} z\,dz + \int_{-\delta_2}^{0} \sigma_2^{(2)} z\,dz$$

$$N_1 = \int_0^{\delta_1} \sigma_1^{(1)} \left(1 - \frac{z}{R}\right) dz + \int_{-\delta_2}^{0} \sigma_1^{(2)} \left(1 - \frac{z}{R}\right) dz \approx \int_0^{\delta_1} \sigma_1^{(1)} dz + \int_{-\delta_2}^{0} \sigma_1^{(2)} dz$$

$$N_2 = \int_0^{\delta_1} \sigma_2^{(1)} dz + \int_{-\delta_2}^{0} \sigma_2^{(2)} dz$$

$$Q_1 = \int_0^{\delta_1} \tau_{13}^{(1)} \left(1 - \frac{z}{R}\right) dz + \int_{-\delta_2}^{0} \tau_{13}^{(2)} \left(1 - \frac{z}{R}\right) dz. \qquad (1.1)$$

Here Index 1 at the right top relates to the internal, and Index 2 to the external layer.

We will examine the deformation of the seam surface. Let u and w be the respective components of the complete displacement of the seam surface in axial and radial directions (Fig. 2).

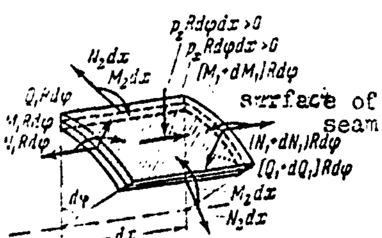

Fig 1

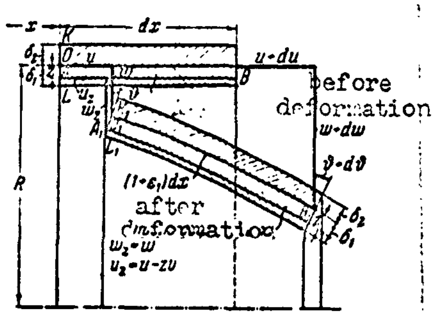

Fig 2

Then the axial relative to deformation of the seam surface equals

$$\varepsilon_1 = \frac{\sqrt{(dx+du)^2+(dw)^2}-dx}{dx} = \sqrt{1+2u'+u'^2+w'^2}-1. \qquad (1.2)$$

The line indicates the derivative by x. Arranging the radicant expression into a series and limiting ourselves to the two first terms of the series, we have

$$\varepsilon_1 = u' + \frac{1}{2}u'^2 + \frac{1}{2}w'^2. \qquad (1.3)$$

Here, the second term of the right side of the expression (1.3) is negligibly small in comparison with the rest. So, during the displacements respective to the wall thickness, the following formula should be used:

$$\varepsilon_1 = u' + \frac{1}{2}w'^2. \qquad (1.4)$$

By a quadratic term during minor displacements we can disregard

$$\varepsilon_1 = u'. \qquad (1.5)$$

The relative annular deformation of the seam surface will be

$$\varepsilon_2 = \frac{2\pi(R-w)-2\pi R}{2\pi R} = -\frac{w}{R}. \qquad (1.6)$$

The incline angle of the normal to the surface of the seam ϑ is related to flexure w by equation of consistency (Fig.2.)

$$\sin\vartheta = \frac{1}{1+\varepsilon_1}w', \qquad (1.7)$$

which at values ϵ_1 are negligible in comparison with the unity and such angles ϑ, in the presence of which angles, substitution of the sine angle by the value of the angle itself is valid and converts into the following:

$$\vartheta = w' \tag{1.8}$$

The obtained formula is valid for small and large displacements of the seam surface only if the indicated conditions are retained.

Relative deformations of the surface of interval z from the seam surface are equal (Fig. 2)

$$\varepsilon_{1z} = \varepsilon_1 - z\varkappa_1, \qquad \varepsilon_{2z} = \varepsilon_2 - z\varkappa_2 \tag{1.9}$$

where

$$\varkappa_1 = w'', \qquad \varkappa_2 = \frac{w}{R^2} \tag{1.10}$$

Normal stresses in the cross-sections are determined by Hooke's law

$$\begin{aligned}
\sigma_1^{(1)} &= \frac{E_1}{1-\mu_1^2}\left[\varepsilon_1 + \mu_1\varepsilon_2 - z(\varkappa_1 + \mu_1\varkappa_2) - (1+\mu_1)\beta_1 t\right] \\
\sigma_2^{(1)} &= \frac{E_1}{1-\mu_1^2}\left[\varepsilon_2 + \mu_1\varepsilon_1 - z(\varkappa_2 + \mu_1\varkappa_1) - (1+\mu_1)\beta_1 t\right] \qquad (0 \leqslant z \leqslant \delta_1) \\
\sigma_1^{(2)} &= \frac{E_2}{1-\mu_2^2}\left[\varepsilon_1 + \mu_2\varepsilon_2 - z(\varkappa_1 + \mu_2\varkappa_2) - (1+\mu_2)\beta_2 t\right] \\
\sigma_2^{(2)} &= \frac{E_2}{1-\mu_2^2}\left[\varepsilon_2 + \mu_2\varepsilon_1 - z(\varkappa_2 + \mu_2\varkappa_1) - (1+\mu_2)\beta_2 t\right] \qquad (-\delta_2 \leqslant z \leqslant 0)
\end{aligned} \tag{1.11}$$

where E_1 and E_2 are moduli of the normal elasticity of the material layers, μ_1 and μ_2 are Poissons ratios of the material, β_1 and β_2 are coefficients of the linear temperature elongation of the material layers, and t is the temperature.

Substituting expression (1.11) into formulas (1.1) we have

$$M_1 = C_1\varepsilon_1 + C_2\varepsilon_2 - D_1\varkappa_1 - D_2\varkappa_2 - g \tag{1.12}$$

$$M_2 = C_2\varepsilon_1 + C_1\varepsilon_2 - D_2\varkappa_1 - D_1\varkappa_2 - g \tag{1.13}$$

$$N_1 = B_1\varepsilon_1 + B_2\varepsilon_2 - C_1\varkappa_1 - C_2\varkappa_2 - f \tag{1.14}$$

$$N_2 = B_2\varepsilon_1 + B_1\varepsilon_2 - C_2\varepsilon_1 - C_1\varkappa_2 - f \tag{1.15}$$

by this

$$
\begin{aligned}
B_1 &= \frac{E_1\delta_1}{1-\mu_1^2} + \frac{E_2\delta_2}{1-\mu_2^2}, \qquad B_2 = \mu_1 \frac{E_1\delta_1}{1-\mu_1^2} + \mu_2 \frac{E_2\delta_2}{1-\mu_2^2} \\
C_1 &= \frac{1}{2}\frac{E_1\delta_1^2}{1-\mu_1^2} - \frac{1}{2}\frac{E_2\delta_2^2}{1-\mu_2^2}, \quad C_2 = \frac{1}{2}\mu_1 \frac{E_1\delta_1^2}{1-\mu_1^2} - \frac{1}{2}\mu_2 \frac{E_2\delta_2^2}{1-\mu_2^2} \\
D_1 &= \frac{1}{3}\frac{E_1\delta_1^3}{1-\mu_1^2} + \frac{1}{3}\frac{E_2\delta_2^3}{1-\mu_2^2}, \quad D_2 = \frac{1}{3}\mu_1 \frac{E_1\delta_1^3}{1-\mu_1^2} + \frac{1}{3}\mu_2 \frac{E_2\delta_2^3}{1-\mu_2^2} \qquad (1.16) \\
f &= \frac{E_1\delta_1 m_1}{1-\mu_1} + \frac{E_2\delta_2 m_2}{1-\mu_2}, \qquad g = \frac{1}{2}\frac{E_1\delta_1^2 n_1}{1-\mu_1} + \frac{1}{2}\frac{E_2\delta_2^2 n_2}{1-\mu_2} \\
m_1 &= \frac{1}{\delta_1}\int_0^{\delta_1} \beta_1 t\,dz,\ m_2 = \frac{1}{\delta_2}\int_{-\delta_2}^{0} \beta_2 t\,dz, \quad n_1 = \frac{2}{\delta_1^2}\int_0^{\delta_1} \beta_1 tz\,dz,\ n_2 = \frac{2}{\delta_2^2}\int_{-\delta_2}^{0} \beta_2 tz\,dz\ ,
\end{aligned}
$$

It is not possible to express transverse force Q_1 by its corresponding deformation.

The equation of equilibrium for an infinitesimal element during fairly big displacements of the seam surface will be (Fig. 1)

$$[N_1(1+u')]' + p_x = 0 \qquad (1.17)$$

$$Q_1' + \frac{M_2}{R^2} + [N_1 w']' + \frac{N_2}{R} + p_z = 0 \qquad (1.18)$$

$$M_1' - Q_1 = 0\ . \qquad (1.19)$$

Here p_x and p_z are components of the external surface load in the direction of axes x and z. From the expression for variation of potential energy of the shell in which the action of the lateral force Q_1 was disregarded

$$
\begin{aligned}
\frac{\delta\Pi}{2\pi R} = &-\int_0^l \left[M_1'' + \frac{M_2}{R^2} + (N_1 w')' + \frac{N_2}{R} + p_z\right]\delta w\,dx - \\
&-\int_0^l \{[N_1(1+u')]' + p_x\}\delta u\,dx + [N_1(1+u')\,\delta u - M_1\delta w' + (M_1' + N_1 w')\,\delta w]_0^l\ , \qquad (1.20)
\end{aligned}
$$

it is evident that the system of the equilibrium equations (1.17) - (1.19) corresponds to the instance when deformations and parameters of the change in curvatures are determined by (1.3), (1.6), (1.10). If ϵ_1 is determined according to (1.4), then the first equilibrium equation (1.17) will be $N_1' + p_x = 0$; from here when $p_x = 0$, we have

$$N_1 = \mathrm{const}. \tag{1.21}$$

In all calculations value x_2 may be considered to equal zero. Then the second and third equilibrium equations (1.18)- (1.19) give

$$M_1'' + (N_1 w')' + \frac{N_2}{R} + p_z = 0.$$

With condition (1.21) we have

$$M_1'' + N_1 w'' + \frac{N_2}{R} + p_z = 0. \tag{1.22}$$

Finally, small displacements may be disregarded by the second term

$$M_1'' + \frac{N_2}{R} + p_z = 0. \tag{1.23}$$

As a result, a complete system of equations is obtained for solving the problem. The number of unknowns equals eleven, the number of equations.

Furthermore, we bear in mind the instance when equation (1.22) is valid. We exclude ϵ_1 from (1.12), (1.13), (1.15), by means of equation (1.14)

$$\epsilon_1 = \frac{1}{B_1}\left[N_1 + f + \frac{B_2}{R} w + C_1 w''\right].$$

Then $(\varkappa_2 = 0)$

$$M_1 = -\frac{1}{R}\frac{C_2B_1 - C_1B_2}{B_1}w - \frac{D_1B_1 - C_1^2}{B_1}w'' + \frac{C_1}{B_1}(N_1 + f) - g \qquad (1.24)$$

$$M_2 = -\frac{1}{R}\frac{C_1B_1 - C_2B_2}{B_1}w - \frac{D_2B_1 - C_1C_2}{B_1}w'' + \frac{C_2}{B_1}(N_1 + f) - g \qquad (1.25)$$

$$N_2 = -\frac{1}{R}\frac{B_1^2 - B_2^2}{B_1}w - \frac{C_2B_1 - C_1B_2}{B_1}w'' + \frac{B_2}{B_1}\left[N_1 + \left(1 - \frac{B_1}{B_2}\right)f\right] \qquad (1.26)$$

We will substitute expressions (1.24) and (1.26) into equation (1.22)

$$w^{IV} + 2aw'' + b^2w = \theta(x) \qquad (1.27)$$

where

$$a = \frac{1}{R}\frac{C_2B_1 - C_1B_2 - {}^1/_2B_1RN_1}{D_1B_1 - C_1^2}, \qquad b^2 = \frac{1}{R^2}\frac{B_1^2 - B_2^2}{D_1B_1 - C_1^2} \qquad (1.28)$$

$$\theta(x) = \frac{B_1}{D_1B_1 - C_1^2}\left\{\frac{C_1}{B_1}(N_1 + f)'' + \frac{1}{R}\frac{B_2}{B_1}\left[N_1 + \left(1 - \frac{B_1}{B_2}\right)f\right] - g'' + p_z\right\} \qquad (1.29)$$

The problem of axially symmetric deformation of an elastic thin-walled bimetallic cylindrical shell under any relationships between the thickness, under different mechanical characteristics of the material layers, and under arbitrary heating along the thickness in the axial direction, is allreduced toward solving equation (1.27).

During the calculation of shell stability, it should be assumed in expression (1.28) that $N_1 = 0$; then equation (1.27) will correspond with the initial equation (1.23).

With $C_1 = C_2 = 0$

$$\mu_1 = \mu_2 = \mu, \qquad E_1\delta_1^2 = E_2\delta_2^2 \qquad (1.30)$$

the calculation is simplified. By this

$$M_1 = -D[w'' + (1+\mu)n], \qquad M_2 = -D[\mu w'' + (1+\mu)n]$$
$$N_1 = B\left[\varepsilon_1 - \mu\frac{w}{R} - (1+\mu)\,m\right], \qquad N_2 = \mu N_1 - \frac{B}{R}(1-\mu^2)(w+mR) \tag{1.31}$$

where

$$B = \frac{\delta\sqrt{E_1E_2}}{1-\mu^2}; \qquad D = \frac{1}{3}\,\frac{E_1E_2\delta^3}{(1-\mu^2)(\sqrt{E_1}+\sqrt{E_2})^2}$$
$$m = \frac{m_1 + m_2\sqrt{E_2/E_1}}{1+\sqrt{E_2/E_1}}, \qquad n = \frac{3}{2}\,\frac{n_1+n_2}{\delta} \tag{1.32}$$

during which δ is the thickness of the shell.

Consequently, instead of equation (1.27) we get

$$w^{IV} - \frac{RN_1}{D}w'' + 4k^4w = \frac{4k^4R}{(1-\mu^2)B}(\mu N_1 + p_2R) - 4k^4Rm - (1+\mu)n'' \tag{1.33}$$

Here

$$k = \sqrt[4]{\frac{(1-\mu^2)B}{4DR^2}} = \frac{\sqrt[4]{3(1-\mu^2)}}{\sqrt{2R\delta}}\,\frac{\sqrt{\sqrt{E_1}+\sqrt{E_2}}}{\sqrt[4]{E_1E_2}}. \tag{1.34}$$

With $E_1 = E_2$, i.e., for monometal we have

$$k = \sqrt[4]{3(1-\mu^2)}\,\frac{1}{\sqrt{R\delta}}. \tag{1.35}$$

In the following paragraphs 2 - 9, the strength of the bimetallic cylindrical shell is investigated; and in paragraphs 10 - 11 , the static and dynamic stability of the shell during axially symmetric deformation due to axial compressive forces.

2. <u>The Integral of the Basic Equation of the Problem.</u> Here, we investigate the instance which may be disregarded by the second term in the equation (1.22). We will pause and find a general solution to the homogeneous equation (1.27).

Substituting $w = \exp sx$, we get a characteristic equation

$$s^4 + 2as^2 + b^2 = 0.$$

From this

$$s = \pm\sqrt{-a \pm \sqrt{a^2 - b^2}} \qquad (2.1)$$

The solution of homogeneous equation (1.27) may be presented as

$$w^* = T_1 \operatorname{ch} \alpha x \cos \beta x + T_2 \operatorname{sh} \alpha x \sin \beta x + T_3 \operatorname{sh} \alpha x \cos \beta x + T_4 \operatorname{ch} \alpha x \sin \beta x \qquad (2.2)$$

where T_1, T_2, T_3, T_4 are arbitrary constants.

$$\alpha = \sqrt{\frac{b - a}{2}}, \qquad \beta = \sqrt{\frac{b + a}{2}} \qquad (2.3)$$

For this, it should be shown that $\alpha^2 < b^2$, i.e., all roots of the characteristic equation (2.1), are complex. First we calculate

$$C_2B_1 - C_1B_2 = \frac{1}{2}(\mu_1 - \mu_2)\frac{E_1E_2\delta_1\delta_2(\delta_1+\delta_2)}{(1-\mu_1^2)(1-\mu_2^2)}$$

$$B_1^2 - B_2^2 = \frac{E_1^2\delta_1^2}{1-\mu_1^2} + \frac{E_2^2\delta_2^2}{1-\mu_2^2} + 2(1-\mu_1\mu_2)\frac{E_1E_2\delta_1\delta_2}{(1-\mu_1^2)(1-\mu_2^2)}$$

$$D_1B_1 - C_1^2 = \frac{1}{12}\frac{E_1^2\delta_1^4}{(1-\mu_1^2)^2} + \frac{1}{12}\frac{E_2^2\delta_2^4}{(1-\mu_2^2)^2} + \frac{1}{6}\frac{E_1E_2\delta_1\delta_2}{(1-\mu_1^2)(1-\mu_2^2)}(2\delta_1^2 + 2\delta_2^2 + 3\delta_1\delta_2).$$

We tabulate the following expression:

$$\begin{aligned}
(a^2-b^2)(D_1B_1-C_1^2)R^2 &= (C_2B_1-B_2C_1)^2 - (B_1^2-B_2^2)(D_1B_1-C_1^2) = \\
&= -\Big\{\frac{1}{12}\frac{E_1^4\delta_1^6}{(1-\mu_1^2)^3} + \frac{1}{12}\frac{E_2^4\delta_2^6}{(1-\mu_2^2)^3} + \frac{1}{12}\frac{E_1^2E_2^2\delta_1^2\delta_2^4}{(1-\mu_1^2)(1-\mu_2^2)^2} + \\
&+ \frac{1}{12}\frac{E_1^2E_2^2\delta_1^4\delta_2^2}{(1-\mu_1^2)^2(1-\mu_2^2)} + \frac{1}{6}\frac{E_1^3E_2\delta_1^3\delta_2}{(1-\mu_1^2)^2(1-\mu_2^2)}(2\delta_1^2+2\delta_2^2+3\delta_1\delta_2) + \\
&+ \frac{1}{6}\frac{E_1E_2^3\delta_1\delta_2^3}{(1-\mu_1^2)(1-\mu_2^2)^2}(2\delta_1^2+2\delta_2^2+3\delta_1\delta_2) + \frac{1}{6}\frac{E_1^3E_2\delta_1^3\delta_2}{(1-\mu_1^2)^2(1-\mu_2^2)}(1-\mu_1\mu_2) + \\
&+ \frac{1}{6}\frac{E_1E_2^3\delta_1\delta_2^3}{(1-\mu_1^2)(1-\mu_2^2)^2}(1-\mu_1\mu_2) + \frac{1}{12}\frac{E_1^2E_2^2\delta_1^2\delta_2^2}{(1-\mu_1^2)^2(1-\mu_2^2)} \times \\
&\times \left[(\delta_1^2+\delta_2^2)(8-3\mu_1^2-3\mu_2^2-2\mu_1\mu_2) + 6\delta_1\delta_2(2-\mu_1^2-\mu_2^2)\right]\Big\}.
\end{aligned}$$

With $\mu_1 < 1$ and $\mu_2 < 1$, multinomial, enclosed in braces, it is always positive; thus

$$(C_2B_1 - B_2C_1)^2 - (B_1^2 - B_2^2)(D_1B_1 - C_1^2) < 0.$$

From this, it follows that $\alpha^2 < b^2$. The general solution of equation (1.27) will be

$$w = w^* + w_0 \qquad (2.4)$$

where w_0 is a particular solution of the heterogeneous equation (1.27).

3. A Cylindrical Shell of Finite Length.

We introduce the designation

$$\Phi_1(x) = \operatorname{ch}\alpha x\cos\beta x, \quad \Phi_2(x) = \operatorname{sh}\alpha x\sin\beta x$$

$$\Phi_3(x) = \operatorname{sh}\alpha x\cos\beta x, \quad \Phi_4(x) = \operatorname{ch}\alpha x\sin\beta x.$$

The derivatives of functions $\Phi_i(x)$ ($i = 1, 2, 3, 4$) along x and the integrals of these functions are indicated below (the value of the argument is omitted)

$$\Phi_1' = \alpha\Phi_3 - \beta\Phi_4, \quad \Phi_2' = \alpha\Phi_4 + \beta\Phi_3$$

$$\Phi_3' = \alpha\Phi_1 - \beta\Phi_2, \quad \Phi_4' = \alpha\Phi_2 + \beta\Phi_1$$

$$\Phi_1'' = (\alpha^2 - \beta^2)\Phi_1 - 2\alpha\beta\Phi_2, \quad \Phi_2'' = (\alpha^2 - \beta^2)\Phi_2 + 2\alpha\beta\Phi_1$$

$$\Phi_3'' = (\alpha^2 - \beta^2)\Phi_3 - 2\alpha\beta\Phi_4, \quad \Phi_4'' = (\alpha^2 - \beta^2)\Phi_4 + 2\alpha\beta\Phi_3$$

$$\Phi_1''' = \alpha(\alpha^2 - 3\beta^2)\Phi_3 + \beta(\beta^2 - 3\alpha^2)\Phi_4, \quad \Phi_2''' = \alpha(\alpha^2 - 3\beta^2)\Phi_4 - \beta(\beta^2 - 3\alpha^2)\Phi_3$$

$$\Phi_3''' = \alpha(\alpha^2 - 3\beta^2)\Phi_1 + \beta(\beta^2 - 3\alpha^2)\Phi_2, \quad \Phi_4''' = \alpha(\alpha^2 - 3\beta^2)\Phi_2 - \beta(\beta^2 - 3\alpha^2)\Phi_1$$

$$\Phi_1^{IV} = (\alpha^4 - 6\alpha^2\beta^2 + \beta^4)\Phi_1 - 4\alpha\beta(\alpha^2 - \beta^2)\Phi_2$$

$$\Phi_2^{IV} = (\alpha^4 - 6\alpha^2\beta^2 + \beta^4)\Phi_2 + 4\alpha\beta(\alpha^2 - \beta^2)\Phi_1$$

$$\Phi_3^{IV} = (\alpha^4 - 6\alpha^2\beta^2 + \beta^4)\Phi_3 - 4\alpha\beta(\alpha^2 - \beta^2)\Phi_4$$

$$\Phi_4^{IV} = (\alpha^4 - 6\alpha^2\beta^2 + \beta^4)\Phi_4 + 4\alpha\beta(\alpha^2 - \beta^2)\Phi_3$$

$$j_1(x) = \int_0^x \Phi_1 dx = \frac{\alpha\Phi_3 + \beta\Phi_4}{\alpha^2 + \beta^2}, \quad j_2(x) = \int_0^x \Phi_2 dx = \frac{\alpha\Phi_4 - \beta\Phi_3}{\alpha^2 + \beta^2}$$

$$j_3(x) = \int_0^x \Phi_3 dx = \frac{\alpha(\Phi_1 - 1) + \beta\Phi_2}{\alpha^2 + \beta^2}, \quad j_4(x) = \int_0^x \Phi_4 dx = \frac{-\beta(\Phi_1 - 1) + \alpha\Phi_2}{\alpha^2 + \beta^2}.$$

The values of functions $\Phi_i(x)$ and their derivatives at zero value of the argument are shown in Table I.

Table I

i	$\Phi_i(0)$	$\Phi_i'(0)$	$\Phi_i''(0)$	$\Phi_i'''(0)$	$\Phi_i^{IV}(0)$
1	1	0	$\alpha^2 - \beta^2$	0	$\alpha^4 - 6\alpha^2\beta^2 + \beta^4$
2	0	0	$2\alpha\beta$	0	$4\alpha\beta(\alpha^2 - \beta^2)$
3	0	α	0	$\alpha(\alpha^2 - 3\beta^2)$	0
4	0	β	0	$-\beta(\beta^2 - 3\alpha^2)$	0

For the forces of (1.12), (1.13), (1.15), (1.19) we obtain the expression

$$M_1 = k_1\Phi_2 T_1 - k_1\Phi_1 T_2 + k_1\Phi_4 T_3 - k_1\Phi_3 T_4 - \frac{1}{R}\,\frac{C_2B_1 - B_2C_1}{B_1}\,w_0 - \frac{D_1B_1 - C_1^2}{B_1}\,w_0'' + \frac{C_1}{B_1}(N_1 + f) - g \tag{3.1}$$

$$M_2 = -(k_2\Phi_1 - k_3\Phi_2)T_1 - (k_2\Phi_2 + k_3\Phi_1)T_2 - (k_2\Phi_3 - k_3\Phi_4)T_3 - \tag{3.2}$$
$$-(k_2\Phi_4 + k_3\Phi_3)T_4 - \frac{C_1B_1 - C_2B_2}{RB_1}w_0 - \frac{D_1B_1 - C_1C_2}{B_1}w_0'' + \frac{C_2}{B_1}(N_1 + f) - g$$

$$N_2 = -(k_4\Phi_1 - k_5\Phi_2)T_1 - (k_4\Phi_2 + k_5\Phi_1)T_2 - (k_4\Phi_3 - k_5\Phi_4)T_3 - \tag{3.3}$$
$$-(k_4\Phi_4 + k_5\Phi_3)T_4 - \frac{B_1^2 - B_2^2}{RB_1}w_0 - \frac{C_2B_1 - C_1B_2}{B_1}w_0'' + \frac{B_2}{B_1}\left[N_1 + \left(1 - \frac{B_1}{B_2}\right)f\right]$$
$$Q_1 = k_1(\beta\Phi_3 + \alpha\Phi_4)T_1 + k_1(\beta\Phi_4 - \alpha\Phi_3)T_2 + k_1(\beta\Phi_1 + \alpha\Phi_2)T_3 +$$
$$+ k_1(\beta\Phi_2 - \alpha\Phi_1)T_4 - \frac{1}{R}\frac{C_2B_1 - C_1B_2}{B_1}w_0' - \frac{D_1B_1 - C_1^2}{B_1}w_0''' + \frac{C_1}{B_1}(f' - p_x) - g'$$

where

$$k_1 = 2\alpha\beta\frac{D_1B_1 - C_1^2}{B_1}, \quad k_2 = \frac{1}{R}\frac{C_1B_1 - C_2B_2}{B_1} + (\alpha^2 - \beta^2)\frac{D_1B_1 - C_1C_2}{B_1} \tag{3.4}$$
$$k_3 = 2\alpha\beta\frac{D_2B_1 - C_1C_2}{B_1}, \quad k_4 = \frac{1}{R}\frac{B_1^2 - B_2^2}{B_1} + (\alpha^2 - \beta^2)\frac{C_2B_1 - C_1B_2}{B_1} \tag{3.5}$$
$$k_5 = 2\alpha\beta\frac{C_2B_1 - C_1B_2}{B_1}.$$

The normal meridianal force N_1 is dependent on either the nature of the problem or the condition of axial displacement, or on the equilibrium equation of a shell part.

Axial displacement is determined by formula (1.5)

$$u = T_5 + \frac{1}{B_1}\Big\{(\alpha H_1\Phi_3 - \beta H_2\Phi_4)T_1 + (\alpha H_2\Phi_4 + \beta H_1\Phi_3)T_2 +$$
$$+\left(\alpha H_2\Phi_1 - \beta H_1\Phi_2 - \frac{\alpha B_2}{(\alpha^2+\beta^2)R}\right)T_3 + \left(\alpha H_2\Phi_2 + \beta H_1\Phi_1 + \right. \tag{3.6}$$
$$\left. + \frac{\beta B_2}{(\alpha^2+\beta^2)R}\right)T_4 + \int_0^x (N_1 + f)\,dx + \frac{B_2}{R}\int_0^x w_0\,dx + C_1w_0'\Big\},$$

where T_5 is the arbitrary constant.

$$H_1 = C_1 + \frac{B_2}{(\alpha^2+\beta^2)R}, \qquad H_2 = C_2 - \frac{B_2}{(\alpha^2+\beta^2)R} \tag{3.7}$$

Equations (3.1) - (3.4) are valid for a thin bimetallic cylindrical shell with arbitrary thickness radios δ_1 and δ_2, elasticity moduli E_1 and E_2, and Poisson's coefficients μ_1 and μ_2. Under these conditions the deformation was assumed to be elastic as well as axially symmetric.

4. <u>A Cylindrical Shell of Infinite Length.</u> The solution of the homogeneous equation (1.27) for a shell of infinite length may be obtained directly from expression (2.2). For this purpose we express the hyperbolic sine and cosine through exponential functions. Since $x \to \infty$, terms containing exp αx tend toward infinity, the constants of integration for a shell of infinite length, with these terms, may be assumed to equal zero. Then

$$w = T_1 A_1(\alpha x, \beta x) + T_2 A_2(\alpha x, \beta x) + w_0 \qquad (4.1)$$

where T_1, T_2 are the new constants of integration, determined from conditions in the investigated edge

$$A_1 = A_1(\alpha x, \beta x) = e^{-\alpha x}\sin\beta x, \qquad A_2 = A_2(\alpha x, \beta x) = e^{-\alpha x}\cos\beta x.$$

The first four derivatives of $A_i(\alpha x, \beta x)$ $(i = 1, 2)$ along x and the integrals of these functions are indicated below (the value of the argument is omitted)

$$A_1' = -\alpha A_1 + \beta A_2, \qquad A_2' = -\beta A_1 - \alpha A_2$$

$$A_1'' = (\alpha^2 - \beta^2) A_1 - 2\alpha\beta A_2, \qquad A_2'' = (\alpha^2 - \beta^2) A_2 + 2\alpha\beta A_1$$

$$A_1''' = -\alpha(\alpha^2 - 3\beta^2) A_1 - \beta(\beta^2 - 3\alpha^2) A_2$$

$$A_2''' = -\alpha(\alpha^2 - 3\beta^2) A_1 - \alpha(\alpha^2 - 3\beta^2) A_2$$

$$A_1^{IV} = (\alpha^4 - 6\alpha^2\beta^2 + \beta^4) A_1 - 4\alpha\beta(\alpha^2 - \beta^2) A_2$$

$$A_2^{IV} = (\alpha^4 - 6\alpha^2\beta^2 + \beta^4) A_2 + 4\alpha\beta(\alpha^2 - \beta^2) A_1$$

$$G_1(x) = \int_0^x A_1\,dx = \frac{\beta(1 - A_2) - \alpha A_1}{\alpha^2 + \beta^2}, \qquad G_2(x) = \int_0^x A_2\,dx = \frac{\alpha(1 - A_2) + \beta A_1}{\alpha^2 + \beta^2}.$$

The values of functions $A_i(\alpha x, \beta x)$ and their derivatives with $x = 0$ are indicated in Table II.

Table II

i	$A_i(0, 0)$	$A_i'(0, 0)$	$A_i''(0, 0)$	$A_i'''(0, 0)$	$A_i^{IV}(0, 0)$
1	0	β	$-2\alpha\beta$	$-\beta(\beta^2 - 3\alpha^2)$	$-4\alpha\beta(\alpha^2 - \beta^2)$
2	1	$-\alpha$	$\alpha^2 - \beta^2$	$-\alpha(\alpha^2 - 3\beta^2)$	$\alpha^4 - 6\alpha^2\beta^2 + \beta^4$

Internal forces and axial displacements for a long shell equal

$$M_1 = k_1 T_1 A_2 - k_1 T_2 A_1 + \frac{1}{R}\frac{C_1 B_2 - B_1 C_2}{B_1} w_0 - \frac{D_1 B_1 - C_1^2}{B_1} w_0'' + \frac{C_1}{B_1}(N_1 + f) - g \tag{4.2}$$

$$M_2 = -(k_2 A_1 - k_3 A_2) T_1 - (k_2 A_2 + k_3 A_1) T_2 - \frac{1}{R}\frac{C_2 B_1 - C_1 B_2}{B_1} w_0 - \frac{D_2 B_1 - C_1 C_2}{B_1} w_0'' + \frac{C_2}{B_1}(N_1 + f) - g \tag{4.3}$$

$$N_2 = -(k_4 A_1 - k_5 A_2) T_1 - (k_4 A_2 + k_5 A_1) T_2 - \frac{1}{R}\frac{B_1^2 - B_2^2}{B_1} w_0 - \frac{C_2 B_1 - C_1 B_2}{B_1} w_0'' + \frac{B_2}{B_1}\left[N_1 + \left(1 - \frac{B_1}{B_2}\right) f\right] \tag{4.4}$$

$$Q_1 = k_1(\alpha A_2 - \beta A_1) T_1 + k_1(\alpha A_1 - \beta A_2) T_2 - \frac{1}{R}\frac{C_2 B_1 - C_1 B_2}{B_1} w_0' - \frac{D_1 B_1 - C_1^2}{B_1} w_0''' + \frac{C_1}{B_1}(f' - p_x) - g' \tag{4.5}$$

$$u = T_3 + \frac{1}{B_1}\left\{\left[-\left(\alpha C_1 + \frac{\beta B_2}{(\alpha^2 + \beta^2) R}\right) A_2 - \left(\beta C_1 + \frac{\alpha B_2}{(\alpha^2 + \beta^2) R}\right) A_1 + \frac{\beta B_2}{(\alpha^2 + \beta^2) R}\right] T_1 + \left[\left(\beta C_1 - \frac{\alpha B_2}{(\alpha^2 + \beta^2) R}\right) A_2 - \left(\alpha C_1 + \frac{\beta B_2}{(\alpha^2 + \beta^2) R}\right) A_1 + \frac{\alpha B_2}{(\alpha^2 + \beta^2) R}\right] T_2 + \int_0^x (N_1 + f)\,dx + \frac{B_2}{R}\int_0^x w_0\,dx + C_1 w_0'\right\}. \tag{4.6}$$

5. A Cylindrical Shell for Which the Condition of Reduction is Applicable.

With $C_1 = C_2 = 0$, the so-called condition of reduction $\mu_1 = \mu_2 = \mu$ and $E_1\delta_1^2 = E_2\delta_2^2$ is valid. From a mechanical viewpoint these conditions satisfy a state, during which the seam surface of the shell becomes a neutral surface during bending. From the mathematical viewpoint the

calculation for the bimetallic she l compares with the calculation for a homogeneous (monometallic) shell, but with several introduced moduli, one of which is the modulus of elasticity during bending and analogous to the known modulus, introduced during the calculation of stability beyond the elasticity boundary of the rectangular cross-section of a rectilinear bar

$$E_k = \frac{4E_1E_2}{(\sqrt{E_1} + \sqrt{E_2})^2},$$

the other is the modulus of elasticity on stretch and is equal to the square root from derivatives of the elasticity moduli of layers $\sqrt{E_1E_2}$.

In accordance with (1.28), $\alpha = 0$, and with (2.3), $\alpha = \beta$. All functions for the investigated instance are received if we assume that $C_1 = C_2 = 0$ in sections 3 and 4.

Table III

i	$F_i'(\xi)$	$F_i''(\xi)$	$F_i'''(\xi)$	$F_i^{IV}(\xi)$	$S_i(\xi)$
1	$-4kF_4(\xi)$	$-4k^2F_3(\xi)$	$-4k^3F_2(\xi)$	$-4k^4F_1(\xi)$	$F_2(\xi)/k$
2	$kF_1(\xi)$	$-4k^2F_4(\xi)$	$-4k^3F_3(\xi)$	$-4k^4F_2(\xi)$	$F_3(\xi)/k$
3	$kF_2(\xi)$	$k^2F_1(\xi)$	$-4k^3F_4(\xi)$	$-4k^4F_3(\xi)$	$F_4(\xi)/k$
4	$kF_3(\xi)$	$k^2F_2(\xi)$	$k^3F_1(\xi)$	$-4k^4F_4(\xi)$	$[1-F_1(\xi)]/4k$

The solution may be presented, also, through the A. N. Krilov functions introduced by him when making calculations for bars on an elastic base /4/.

Table IV

i	$F_i(0)$	$F_i'(0)$	$F_i''(0)$	$F_i'''(0)$	$F_i^{IV}(0)$
1	1	0	0	0	$-4k^4$
2	0	k	0	0	0
3	0	0	k^2	0	0
4	0	0	0	k^3	0

$$F_1(\xi) = \operatorname{ch}\xi\cos\xi \qquad (\xi = kx)$$

$$F_2(\xi) = \frac{1}{2}(\operatorname{ch}\xi\sin\xi + \operatorname{sh}\xi\cos\xi)$$

$$F_3(\xi) = \frac{1}{2}\operatorname{sh}\xi\sin\xi$$

$$F_4(\xi) = \frac{1}{4}\cdot(\operatorname{ch}\xi\sin\xi - \operatorname{sh}\xi\cos\xi)$$

The derivatives of A.N. Krilov's functions along x are shown in Table III. The value of A.N. Krilo'v functions, as well as of their derivatives when $x = 0$, are shown in Table IV; by which

$$S_i(\xi) = \int_0^x F(\xi)\,dx .$$

A general solution of equation (1.33) will be

$$w = T_1F_1(\xi) + T_2F_2(\xi) + T_3F_3(\xi) + T_4F_4(\xi) + w_0 \qquad (5.1)$$

where T_1, T_2, T_3, T_4 are the new constants of integration.

The expressions for forces and the angle of inclination of the normal will be

$$\begin{aligned}
M_1 &= k^2D\,[4T_1F_3(\xi) + 4T_2F_4(\xi) - T_3F_1(\xi) - T_4F_2(\xi)] - D[w_0'' + (1+\mu)n] \\
M_2 &= \mu k^2D[4T_1F_3(\xi) + 4T_2F_4(\xi) - T_3F_1(\xi) - T_4F_2(\xi)] - D\,[\mu w_0'' + (1+\mu)n] \\
Q_1 &= k^3D\,[4T_1F_2(\xi) + 4T_2F_3(\xi) + 4T_3F_4(\xi) - T_4F_1(\xi)] - D[w_0''' + (1+\mu)\,n'] \\
N_2 &= \mu N_1 - \frac{(1-\mu^2)\,B}{R}\,[T_1F_1(\xi) + T_2F_2(\xi) + T_3F_3(\xi) + T_4F_4(\xi) + w_0 + mR] \qquad (5.2) \\
\vartheta &= k\,[-4T_1F_4(\xi) + T_2F_1(\xi) + T_3F_2(\xi) + T_4F_3(\xi)] + w_0'
\end{aligned}$$

But for a Cylinder of Infinite length we will find /5/

$$\begin{aligned}
M_1 &= 2k^2D[T_1A_2(\xi) - T_2A_1(\xi)] - D\,[w_0'' + (1+\mu)n] \\
M_2 &= 2\mu k^2D[T_1A_2(\xi) - T_2A_1(\xi)] - D[\mu w_0'' + (1+\mu)n] \\
Q_1 &= -2k^3D\,[T_1A_3(\xi) + T_2A_4(\xi)] - D\,[w_0''' + (1+\mu)n'] \\
N_2 &= \mu N_1 - \frac{(1-\mu^2)\,B}{R}\,[T_1A_1(\xi) + T_2A_2(\xi) + w_0 + mR] \qquad (5.3) \\
w &= T_1A_1(\xi) + T_2A_2(\xi) + w_0, \qquad \vartheta = k[T_1A_4(\xi) - T_2A_3(\xi)] + w_0'
\end{aligned}$$

Here

$$A_1(\xi) = e^{-\xi}\sin\xi, \qquad A_3(\xi) = e^{-\xi}(\sin\xi + \cos\xi)$$
$$A_2(\xi) = e^{-\xi}\cos\xi, \qquad A_4(\xi) = e^{-\xi}(\cos\xi - \sin\xi)$$

by T_1, T_2, certain other arbitrary constants are indicated here.

Table V

i	$A_i'(\xi)$	$A_i''(\xi)$	$A_i'''(\xi)$	$A_i^{IV}(\xi)$	$X_i(\xi)$
1	$kA_4(\xi)$	$-2k^2A_3(\xi)$	$2k^3A_2(\xi)$	$-4k^4A_1(\xi)$	$[1-A_2(\xi)]/2k$
2	$-kA_3(\xi)$	$2k^2A_1(\xi)$	$2k^3A_4(\xi)$	$-4k^4A_2(\xi)$	$[1-A_4(\xi)]/2k$
3	$-2kA_1(\xi)$	$-2k^2A_4(\xi)$	$4k^3A_2(\xi)$	$-4k^4A_3(\xi)$	$[1-A_3(\xi)]/k$
4	$-2kA_2(\xi)$	$2k^2A_3(\xi)$	$-4k^3A_1(\xi)$	$-4k^4A_4(\xi)$	$A_1(\xi)/k$

The derivatives of functions $A_i(\xi)$ ($i = 1,\ldots 4$) along x are shown in Table V. The value of these functions, as well as of the derivatives, when $\xi = 0$ are given in Table VI.

Also presented are the values of the integrals

$$X_i = \int_0^x A_i(\xi)\, dx$$

Table VI

i	$A_i(0)$	$A_i'(0)$	$A_i''(0)$	$A_i'''(0)$	$A_i^{IV}(0)$
1	0	k	$-2k^2$	$2k^3$	0
2	1	$-k$	0	$2k^3$	$-4k^4$
3	1	0	$-2k^2$	$4k^3$	$-4k^4$
4	1	$-2k$	$2k^2$	0	$-4k^4$

6. Boundary Conditions. Boundary conditions are in brackets in formula (1.20). There are a total of 6 boundary conditions- three on each edge. They permit determination of constants N_1, T_1, T_2, T_3, T_4, T_5.

The form adopted for solving the problem brings an accord of the boundary conditions for homogeneous and bimetallic shells.

Thus, as an example for the free edge

$$N_1 = M_1 = Q_1 = 0.$$

For the rigidly fixed edge

$$u = v = w = 0.$$

7. The Determination of Normal Stresses. Normal stresses in layers of the shell are found by formulas (1.11), where ϵ_1, ϵ_2, x_1, x_2 are determined through internal forces from (1.12) - (1.15) by formulas:

$$\varepsilon_1 = \frac{\Delta_1}{\Delta}, \qquad \varepsilon_2 = \frac{\Delta_2}{\Delta}, \qquad \varkappa_1 = \frac{\Delta_3}{\Delta}, \qquad \varkappa_2 = \frac{\Delta_4}{\Delta}. \tag{7.1}$$

Here

$$\begin{aligned}
\Delta_1 &= (N_1 + f) E_1 + (N_2 + f) E_2 + (M_1 + g) E_3 + (M_2 + g) E_4 \\
\Delta_2 &= (N_1 + f) E_2 + (N_2 + f) E_1 + (M_1 + g) E_4 + (M_2 + g) E_3 \\
\Delta_3 &= -(N_1 + f) E_3 - (N_2 + f) E_4 + (M_1 + g) E_5 + (M_2 + g) E_6 \\
\Delta_4 &= -(N_1 + f) E_4 - (N_2 + f) E_3 + (M_1 + g) E_6 + (M_2 + g) E_5 \\
\Delta &= (B_1^2 - B_2^2)(D_1^2 - D_2^2) - 2(B_2D_2 + B_1D_1)(C_1^2 + C_2^2) + \\
&\quad + 4C_1C_2(B_1D_2 + B_2D_1) + (C_1^2 - C_2^2)^2 \\
E_1 &= B_1(D_1^2 - D_2^2) + 2C_1C_2D_2 - D_1(C_1^2 + C_2^2) \\
E_2 &= -B_2(D_1^2 - D_2^2) + 2C_1C_2D_1 - D_2(C_1^2 + C_2^2) \\
E_3 &= -B_2(C_1D_2 - C_2D_1) + C_1(C_1^2 - C_2^2) + B_1(C_2D_2 - C_1D_1) \\
E_4 &= -B_2(-C_1D_1 + C_2D_2) - C_1(-B_1D_2 + C_1C_2) + C_2(-B_1D_1 + C_2^2) \\
E_5 &= B_1(C_1^2 + C_2^2) - 2B_2C_1C_2 - D_1(B_1^2 - B_2^2) \\
E_6 &= B_2(C_1^2 + C_2^2) - 2B_1C_1C_2 + D_2(B_1^2 - B_2^2)
\end{aligned} \tag{7.2}$$

If Poisson's ratios of the material layers are identical $\mu_1 = \mu_2 = \mu$, then the introduced expressions for ϵ_1, ϵ_2, x_1, x_2 become simplified, as

$$C_2 = \mu C_1, \qquad D_2 = \mu D_1, \qquad B_2 = \mu B_1.$$

By this

$$\begin{aligned}
\varepsilon_1 &= \frac{[N_1 - \mu N_2 + (1-\mu) f] D_1 - [M_1 - \mu M_2 + (1-\mu) g] C_1}{(1-\mu^2)(D_1B_1 - C_1^2)} \\
\varepsilon_2 &= \frac{[N_2 - \mu N_1 + (1-\mu) f] D_1 - [M_2 - \mu M_1 + (1-\mu) g] C_1}{(1-\mu^2)(D_1B_1 - C_1^2)} \\
\varkappa_1 &= \frac{[N_1 - \mu N_2 + (1-\mu) f] C_1 - [M_1 - \mu M_2 + (1-\mu) g] B_1}{(1-\mu^2)(D_1B - C_1^2)} \\
\varkappa_2 &= \frac{[N_2 - \mu N_1 + (1-\mu) f] C_1 - [M_2 - \mu M_1 + (1-\mu) g] B_1}{(1-\mu^2)(D_1B_1 - C_1^2)}
\end{aligned} \tag{7.3}$$

Thus, as an example for the free edge

$$N_1 = M_1 = Q_1 = 0.$$

For the rigidly fixed edge

$$u = v = w = 0.$$

7. <u>The Determination of Normal Stresses.</u> Normal stresses in layers of the shell are found by formulas (1.11), where ϵ_1, ϵ_2, x_1, x_2 are determined through internal forces from (1.12) - (1.15) by formulas:

$$\epsilon_1 = \frac{\Delta_1}{\Delta}, \qquad \epsilon_2 = \frac{\Delta_2}{\Delta}, \qquad \varkappa_1 = \frac{\Delta_3}{\Delta}, \qquad \varkappa_2 = \frac{\Delta_4}{\Delta}. \tag{7.1}$$

Here

$$\begin{aligned}
\Delta_1 &= (N_1 + f) E_1 + (N_2 + f) E_2 + (M_1 + g) E_3 + (M_2 + g) E_4 \\
\Delta_2 &= (N_1 + f) E_2 + (N_2 + f) E_1 + (M_1 + g) E_4 + (M_2 + g) E_3 \\
\Delta_3 &= -(N_1 + f) E_3 - (N_2 + f) E_4 + (M_1 + g) E_5 + (M_2 + g) E_6 \\
\Delta_4 &= -(N_1 + f) E_4 - (N_2 + f) E_3 + (M_1 + g) E_6 + (M_2 + g) E_5 \\
\Delta &= (B_1^2 - B_2^2)(D_1^2 - D_2^2) - 2(B_2 D_2 + B_1 D_1)(C_1^2 + C_2^2) + \\
&\quad + 4C_1C_2(B_1D_2 + B_2D_1) + (C_1^2 - C_2^2)^2 \\
E_1 &= B_1(D_1^2 - D_2^2) + 2C_1C_2D_2 - D_1(C_1^2 + C_2^2) \\
E_2 &= -B_2(D_1^2 - D_2^2) + 2C_1C_2D_1 - D_2(C_1^2 + C_2^2) \\
E_3 &= -B_2(C_1D_2 - C_2D_1) + C_1(C_1^2 - C_2^2) + B_1(C_2D_2 - C_1D_1) \\
E_4 &= -B_2(-C_1D_1 + C_2D_2) - C_1(-B_1D_2 + C_1C_2) + C_2(-B_1D_1 + C_2^2) \\
E_5 &= B_1(C_1^2 + C_2^2) - 2B_2C_1C_2 - D_1(B_1^2 - B_2^2) \\
E_6 &= B_2(C_1^2 + C_2^2) - 2B_1C_1C_2 + D_2(B_1^2 - B_2^2)
\end{aligned} \tag{7.2}$$

If Poisson's ratios of the material layers are identical $\mu_1 = \mu_2 = \mu$, then the introduced expressions for ϵ_1, ϵ_2, x_1, x_2 become simplified, as

$$C_2 = \mu C_1, \qquad D_2 = \mu D_1, \qquad B_2 = \mu B_1.$$

By this

$$\begin{aligned}
\epsilon_1 &= \frac{[N_1 - \mu N_2 + (1-\mu) f] D_1 - [M_1 - \mu M_2 + (1-\mu) g] C_1}{(1-\mu^2)(D_1B_1 - C_1^2)} \\
\epsilon_2 &= \frac{[N_2 - \mu N_1 + (1-\mu) f] D_1 - [M_2 - \mu M_1 + (1-\mu) g] C_1}{(1-\mu^2)(D_1B_1 - C_1^2)} \\
\varkappa_1 &= \frac{[N_1 - \mu N_2 + (1-\mu) f] C_1 - [M_1 - \mu M_2 + (1-\mu) g] B_1}{(1-\mu^2)(D_1B - C_1^2)} \\
\varkappa_2 &= \frac{[N_2 - \mu N_1 + (1-\mu) f] C_1 - [M_2 - \mu M_1 + (1-\mu) g] B_1}{(1-\mu^2)(D_1B_1 - C_1^2)}
\end{aligned} \tag{7.3}$$

The following expressions are necessary for calculating stresses during $\mu_1 = \mu_2$:

$$\varepsilon_1 + \mu\varepsilon_2 = \frac{N_1 D_1 - M_1 C_1 + f D_1 - g C_1}{D_1 B_1 - C_1^2}, \qquad \varepsilon_2 + \mu\varepsilon_1 = \frac{N_2 D_1 - M_2 C_1 + f D_1 - g C_1}{D_1 B_1 - C_1^2}$$
$$\varkappa_1 + \mu\varkappa_2 = \frac{N_1 C_1 - M_1 B_1 + f C_1 - g B_1}{D_1 B_1 - C_1^2}, \qquad \varkappa_2 + \mu\varkappa_1 = \frac{N_2 C_1 - M_2 B_1 + f C_1 - g B_1}{D_1 B_1 - C_1^2}. \tag{7.4}$$

If the condition of reduction is valid, then in formulas (7.3) and (7.4) it should be assumed that $C_1 = 0$. Then

$$\varepsilon_1 = \frac{N_1 - \mu N_2}{(1-\mu^2) B} + m, \qquad \varepsilon_2 = \frac{N_2 - \mu N_1}{(1-\mu^2) B} + m$$
$$\varkappa_1 = -\frac{M_1 - \mu M_2}{(1-\mu^2) D} - n, \qquad \varkappa_2 = -\frac{M_2 - \mu M_1}{(1-\mu^2) D} - n.$$

We go over this last case in more detail. Substitution of the obtained values into equations (1.11) gives

$$\begin{aligned}
\sigma_1^{(1)} &= \sqrt{\frac{E_1}{E_2}}\,\frac{N_1}{\delta} + z\frac{3M_1}{\delta^2}\left(1+\sqrt{\frac{E_1}{E_2}}\right)^2 + \frac{E_1}{1-\mu}(m-\beta_1 t+nz) \\
\sigma_2^{(1)} &= \sqrt{\frac{E_1}{E_2}}\,\frac{N_2}{\delta} + z\frac{3M_2}{\delta^2}\left(1+\sqrt{\frac{E_1}{E_2}}\right)^2 + \frac{E_1}{1-\mu}(m-\beta_1 t+nz)
\end{aligned} \qquad (0 \leqslant z \leqslant \delta_1)$$

$$\begin{aligned}
\sigma_1^{(2)} &= \sqrt{\frac{E_2}{E_1}}\,\frac{N_1}{\delta} + z\frac{3M_1}{\delta^2}\left(1+\sqrt{\frac{E_2}{E_1}}\right)^2 + \frac{E_2}{1-\mu}(m-\beta_2 t+nz) \\
\sigma_2^{(2)} &= \sqrt{\frac{E_2}{E_1}}\,\frac{N_2}{\delta} + z\frac{3M_2}{\delta^2}\left(1+\sqrt{\frac{E_2}{E_1}}\right)^2 + \frac{E_2}{1-\mu}(m-\beta_2 t+nz).
\end{aligned} \qquad (-\delta_2 \leqslant z \leqslant 0) \tag{7.5}$$

If it is assumed that $z = 0$ in the above expressions, then we obtain the surface stresses of the shell joint. The second terms of each of the four expressions disappear. This means that at $C_1 = C_2 = 0$, the seam surface assumes the role of a neutral surface.

Assuming that in (7.5), $z = 0$ and $z = \delta_1$, we obtain stresses along the boundaries of the internal layer

$$
\begin{aligned}
\left[\sigma_1^{(1)}\right]_{z=0} &= \sqrt{\frac{E_1}{E_2}}\,\frac{N_1}{\delta} + \frac{E_1}{1-\mu}\left[m - \beta_1 t(0)\right] \\
\left[\sigma_2^{(1)}\right]_{z=0} &= \sqrt{\frac{E_1}{E_2}}\,\frac{N_2}{\delta} + \frac{E_1}{1-\mu}\left[m - \beta_1 t(0)\right] \\
\left[\sigma_1^{(1)}\right]_{z=\delta_1} &= \sqrt{\frac{E_1}{E_2}}\,\frac{N_1}{\delta} + 3\,\frac{M_1}{\delta^2}\left(1+\sqrt{\frac{E_1}{E_2}}\right) + \frac{E_1}{1-\mu}\left[m - \beta_1 t(\delta_1) + n\delta_1\right] \\
\left[\sigma_2^{(1)}\right]_{z=\delta_1} &= \sqrt{\frac{E_1}{E_2}}\,\frac{N_2}{\delta} + 3\,\frac{M_2}{\delta^2}\left(1+\sqrt{\frac{E_1}{E_2}}\right) + \frac{E_1}{1-\mu}\left[m - \beta_1 t(\delta_1) + n\delta_1\right].
\end{aligned}
\tag{7.6}
$$

We obtain the stresses along the edges of the external layer by assuming that in (7.5) $z = 0$ and $z = -\delta_2$

$$
\begin{aligned}
\left[\sigma_1^{(2)}\right]_{z=0} &= \sqrt{\frac{E_2}{E_1}}\,\frac{N_1}{\delta} + \frac{E_2}{1-\mu}\left[m - \beta_2 t(0)\right] \\
\left[\sigma_2^{(2)}\right]_{z=0} &= \sqrt{\frac{E_2}{E_1}}\,\frac{N_2}{\delta} + \frac{E_2}{1-\mu}\left[m - \beta_2 t(0)\right] \\
\left[\sigma_1^{(2)}\right]_{z=-\delta_2} &= \sqrt{\frac{E_2}{E_1}}\,\frac{N_1}{\delta} - 3\,\frac{M_1}{\delta^2}\left(1+\sqrt{\frac{E_2}{E_1}}\right) + \frac{E_2}{1-\mu}\left[m - \beta_2 t(-\delta_2) - n\delta_2\right] \\
\left[\sigma_2^{(2)}\right]_{z=-\delta_2} &= \sqrt{\frac{E_2}{E_1}}\,\frac{N_2}{\delta} - 3\,\frac{M_2}{\delta^2}\left(1+\sqrt{\frac{E_2}{E_1}}\right) + \frac{E_2}{1-\mu}\left[m - \beta_2 t(-\delta_2) - n\delta_2\right].
\end{aligned}
\tag{7.7}
$$

For determining temperature terms, we find in the expression of stresses the expressions m and n as the adopted laws of temperature change.

Let the temperature along the thickness of each layer be constant. In the internal layer the temperature difference before and after deformation is equal to Δt_1, and in the external layer Δt_2. By this from (1.16)

$$
m_1 = n_1 = \beta_1 \Delta t_1, \qquad m_2 = -n_2 = \beta_2 \Delta t_2 ;
$$

so, by (1.32)

$$
m = \frac{\beta_1 \Delta t_1 + \beta_2 \Delta t_2 \sqrt{E_2/E_1}}{1+\sqrt{E_2/E_1}}, \qquad n = \frac{3}{2}\,\frac{\beta_1 \Delta t_1 - \beta_2 \Delta t_2}{\delta}. \tag{7.8}
$$

With $E_1 = E_2$, $\beta_1 = \beta_2 = \beta$, $\Delta t_1 = \Delta t_2 = \Delta t$, i.e., for monometal we get

$$m = \beta \Delta t, \qquad n = 0. \tag{7.9}$$

Let the temperature change linearly along the thickness of the wall:

$$t = t_2 + \frac{t_1 - t_2}{\delta}(\delta_2 + z) = \frac{t_2\delta_1 + t_1\delta_2}{\delta} + \frac{t_1 - t_2}{\delta} z$$

where t_1 is the temperature of the inner layer at $z = \delta_1$, and t_2 is the temperature of the outer layer at $z = \delta_2$. Here

$$m_1 = \frac{\beta_1}{2\delta}[t_2\delta_1 + t_1(2\delta_2 + \delta_1)], \qquad m_2 = \frac{\beta_2}{2\delta}[t_1\delta_2 + t_2(2\delta_1 + \delta_2)]$$
$$n_1 = \frac{\beta_1}{3\delta}[t_2\delta_1 + t_1(3\delta_2 + 2\delta_1)], \qquad n_2 = -\frac{\beta_2}{3\delta}[t_1\delta_2 + t_2(2\delta_2 + 3\delta_1)].$$

These expressions with $C_1 = C_2 = 0$ convert to the following:

$$m_1 = \frac{\beta_1[t_1(2 + \sqrt{E_2/E_1}) + t_2\sqrt{E_2/E_1}]}{2(1 + \sqrt{E_2/E_1})}, \quad m_2 = \frac{\beta_2[t_1 + t_2(1 + 2\sqrt{E_2/E_1})]}{2(1 + \sqrt{E_2/E_1})}$$
$$n_1 = \frac{\beta_1[t_1(3 + 2\sqrt{E_2/E_1}) + t_2\sqrt{E_2/E_1}]}{3(1 + \sqrt{E_2/E_1})}, \quad n_2 = -\frac{\beta_2[t_1 + t_2(2 + 3\sqrt{E_2/E_1})]}{3(1 + \sqrt{E_2/E_1})}$$

Then

$$m = \frac{t_1[\beta_1(2 + \sqrt{E_2/E_1}) + \beta_2\sqrt{E_2/E_1}] + t_2[\beta_1\sqrt{E_2/E_1} + \beta_2(2E_2/E_1 + \sqrt{E_2/E_1})]}{2(1 + \sqrt{E_2/E_1})^2}$$
$$n = \frac{t_1[\beta_1(3 + 2\sqrt{E_2/E_1}) - \beta_2] + t_2[\beta_1\sqrt{E_2/E_1} - \beta_2(2 + 3\sqrt{E_2/E_1})]}{2\delta(1 + \sqrt{E_2/E_1})} \tag{7.10}$$

For monometal $E_1 = E_2 = E$, $\beta_1 = \beta_2 = \beta$ and

$$m = \beta\frac{t_1 + t_2}{2}, \qquad n = \beta\frac{t_1 - t_2}{\delta}. \tag{7.11}$$

a. Constant Temperature Along the Thickness of Each Layer. Substituting expressions of m and n along (7.8) on the right side of equation (7.5) and omitting the terms representing internal forces, we will obtain:

$$\sigma_1^{(1)} = \sigma_2^{(1)} = \frac{\beta_2\Delta t_2 - \beta_1\Delta t_1}{1-\mu}\left[\frac{E_1\sqrt{E_2}}{\sqrt{E_1}+\sqrt{E_2}} - \frac{3}{2}E_1\frac{z}{\delta}\right] \quad (0 < z < \delta_1) \tag{7.12}$$
$$\sigma_1^{(2)} = \sigma_2^{(2)} = -\frac{\beta_2\Delta t_2 - \beta_1\Delta t_1}{1-\mu}\left[\frac{E_2\sqrt{E_1}}{\sqrt{E_1}+\sqrt{E_2}} + \frac{3}{2}E_2\frac{z}{\delta}\right] \quad (-\delta_2 < z < 0).$$

Stresses along the boundaries of each layer will be

$$\left[\sigma_1^{(1)}\right]_{z=0} = \left[\sigma_2^{(1)}\right]_{z=0} = \frac{\beta_2\Delta t_2 - \beta_1\Delta t_1}{1-\mu}\,\frac{E_1\sqrt{E_2}}{\sqrt{E_1}+\sqrt{E_2}}$$
$$\left[\sigma_1^{(1)}\right]_{z=\delta_1} = \left[\sigma_2^{(1)}\right]_{z=\delta_1} = -\frac{1}{2}\,\frac{\beta_2\Delta t_2 - \beta_1\Delta t_1}{1-\mu}\,\frac{E_1\sqrt{E_2}}{\sqrt{E_1}+\sqrt{E_2}} \tag{7.13}$$
$$\left[\sigma_1^{(2)}\right]_{z=0} = \left[\sigma_2^{(2)}\right]_{z=0} = -\frac{\beta_2\Delta t_2 - \beta_1\Delta t_1}{1-\mu}\,\frac{E_2\sqrt{E_1}}{\sqrt{E_1}+\sqrt{E_2}}$$
$$\left[\sigma_1^{(2)}\right]_{z=-\delta_2} = \left[\sigma_2^{(2)}\right]_{z=-\delta_2} = \frac{1}{2}\,\frac{\beta_2\Delta t_2 - \beta_1\Delta t_1}{1-\mu}\,\frac{E_2\sqrt{E_1}}{\sqrt{E_1}+\sqrt{E_2}}.$$

From relationships

$$\frac{\left[\sigma_1^{(1)}\right]_{z=0}}{\left[\sigma_1^{(1)}\right]_{z=\delta_1}} = \frac{\left[\sigma_2^{(1)}\right]_{z=0}}{\left[\sigma_2^{(1)}\right]_{z=\delta_1}} = -2, \qquad \frac{\left[\sigma_1^{(2)}\right]_{z=0}}{\left[\sigma_1^{(2)}\right]_{z=-\delta_2}} = \frac{\left[\sigma_2^{(2)}\right]_{z=0}}{\left[\sigma_2^{(2)}\right]_{z=-\delta_2}} = -2$$

it is evident that when $C_1 = C_2 = 0$, the diagrams of the temperature terms of the stresses are always similar during uniform heating along the thickness of each layer, and the state of the neutral layers during pure thermal deformation ($N_1 = N_2 = M_1 = M_2 = 0$) is always the same ($z = +2\delta_1/3$ and $z = -2\delta_2/3$).

b. The Linear Flow of Temperature Change along the Thickness of the Wall. The temperature terms of the expressions for stresses (7.5) appear as

$$\sigma_1^{(1)} = \sigma_2^{(1)} =$$

$$= \frac{\beta_2 - \beta_1}{2(1-\mu)} \frac{t_1\sqrt{E_1E_2} + t_2(2E_2 + \sqrt{E_1E_2}) - \frac{z}{\delta}(1 + \sqrt{E_2/E_1})\,[t_1E_1 + t_2(2E_1 + 3\sqrt{E_1E_2})]}{(1 + \sqrt{E_2/E_1})^2} \tag{7.14}$$

$$\sigma_1^{(2)} = \sigma_2^{(2)} =$$

$$= -\frac{\beta_2 - \beta_1}{2(1-\mu)} \frac{t_1(2E_1 + \sqrt{E_1E_2}) + t_2\sqrt{E_1E_2} + \frac{z}{\delta}(1 + \sqrt{E_2/E_1})\,[t_1(3E_1 + 2\sqrt{E_1E_2})}{(1 + \sqrt{E_1/E_2})^2} -$$

$$- \frac{t_2\sqrt{E_1E_2}]}{(1 + \sqrt{E_1/E_2})^2}.$$

From here

$$\begin{aligned}
\left[\sigma_1^{(1)}\right]_{z=0} &= \left[\sigma_2^{(1)}\right]_{z=0} = \frac{\beta_2 - \beta_1}{2(1-\mu)} \frac{t_1\sqrt{E_1E_2} + t_2(2E_2 + \sqrt{E_1E_2})}{(1 + \sqrt{E_2/E_1})^2}, \\
\left[\sigma_1^{(1)}\right]_{z=\delta_1} &= \left[\sigma_2^{(1)}\right]_{z=\delta_1} = -\frac{\beta_2 - \beta_1}{2(1-\mu)} \frac{E_2 + \sqrt{E_1E_2}}{(1 + \sqrt{E_2/E_1})^2} t_2 \\
\left[\sigma_1^{(2)}\right]_{z=0} &= \left[\sigma_2^{(2)}\right]_{z=0} = -\frac{\beta_2 - \beta_1}{2(1-\mu)} \frac{t_1(2E_1 + \sqrt{E_1E_2}) + t_2\sqrt{E_1E_2}}{(1 + \sqrt{E_1/E_2})^2} \\
\left[\sigma_1^{(2)}\right]_{z=-\delta_2} &= \left[\sigma_2^{(2)}\right]_{z=-\delta_2} = \frac{\beta_2 - \beta_1}{2(1-\mu)} \frac{E_1 + \sqrt{E_1E_2}}{(1 + \sqrt{E_1/E_2})^2} t_1.
\end{aligned} \tag{7.15}$$

For a homogeneous shell the normal stresses are determined by formulas

$$\begin{aligned}
\sigma_1 &= \frac{N_1}{\delta} + \frac{12M_1z}{\delta^3} + \frac{E}{1-\mu}(m - \beta t + nz) \\
\sigma_2 &= \frac{N_2}{\delta} + \frac{12M_2z}{\delta^3} + \frac{E}{1-\mu}(m - \beta t + nz).
\end{aligned} \tag{7.16}$$

During the linear law of temperature change along the thickness of the wall

$$\sigma_1 = \frac{N_1}{\delta} + \frac{12M_1z}{\delta^3}, \qquad \sigma_2 = \frac{N_2}{\delta} + \frac{12M_2z}{\delta^3}. \tag{7.17}$$

At this stage the stresses along the surface of the shell equal to

$$\sigma_1 = \frac{N_1}{\delta} \mp \frac{6M_1}{\delta^2}, \qquad \sigma_2 = \frac{N_2}{\delta} \mp \frac{6M_2}{\delta^2}. \tag{7.18}$$

In formulas (7.18) the top symbol pertains to points of the outer surface, and the bottom symbol to points of the inner surface.

8. Calculation of Long Cylindrical Shells.

a. A Long Cylindrical Shell with a Fastened Edge during Uniform Heating along the Meridian and with Uniform Pressure.

Letting $p_z = p = \text{const}$, $t = t(z)$, the individual solution of the heterogeneous equation (1.27) will be

$$w_0 = \frac{1}{4\beta^4}\frac{B_1}{D_1B_1 - C_1^2}\left\{\frac{1}{R}\frac{B_1}{B_1}\left[N_1 + \left(1 - \frac{B_1}{B_2}\right)f\right] + p\right\} \qquad (8.1)$$

If the shell is exposed to the action of the axial tensile force, the resultant of which in each section equals N, then

$$N_1 = \frac{N}{2\pi R}.$$

By selecting the origin of reading x of the fastened edge for determination of constants T_1 and T_2, we have conditions $w = w' = 0$ with $x = 0$. From here

$$T_1 = -\frac{\alpha}{\beta}w_0, \qquad T_2 = -w_0$$
$$w = w_0\left[1 - \frac{\alpha}{\beta}A_1(\alpha x, \beta x) - A_2(\alpha x, \beta x)\right]. \qquad (8.2)$$

Forces are determined from expressions (4.2) - (4.5)

$$M_1 = w_0\left\{k_1 A_1(\alpha x, \beta x) - \frac{\alpha}{\beta}k_1 A_2(\alpha x, \beta x) - \frac{C_2B_1 - C_1B_2}{B_1R}\right\} + \frac{C_1}{B_1}(N_1 + f) - g$$
$$M_2 = w_0\left\{\left(k_3 + \frac{\alpha}{\beta}k_2\right)A_1(\alpha x, \beta x) + \left(k_2 - \frac{\alpha}{\beta}k_3\right)A_2(\alpha x, \beta x) - \frac{C_1B_1 - C_2B_2}{B_1R}\right\} + $$
$$+ \frac{C_2}{B_1}(N_1 + f) - g$$
$$Q_1 = -\frac{w_0}{\beta}(\alpha^2 - \beta^2)k_1 A_2(\alpha x, \beta x) \qquad (8.3)$$
$$N_2 = w_0\left\{\left(k_3 + \frac{\alpha}{\beta}k_4\right)A_1(\alpha x, \beta x) + \left(k_4 - \frac{\alpha}{\beta}k_3\right)A_2(\alpha x, \beta x)\right\} - pR.$$

If the heating along the thickness of the wall changes in accordance with the linear law but the shell is homogeneous, then from (8.2) and (8.3) we get (p = 0)

$$
\begin{aligned}
&w = -R\beta\frac{t_1+t_2}{2}[1-A_3(\xi)], \qquad \vartheta = -2R\beta k\frac{t_1+t_2}{2}A_1(\xi)\\
&M_1 = \frac{E\delta^2\beta}{12(1-\mu)}\left[2\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\frac{t_1+t_2}{2}A_4(\xi)-(t_1-t_2)\right]\\
&M_2 = \frac{E\delta^2\beta}{12(1-\mu)}\left[2\mu\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\frac{t_1+t_2}{2}A_4(\xi)-(t_1-t_2)\right]\\
&Q_1 = -4k^3DR\beta\frac{t_1+t_2}{2}A_2(\xi), \qquad N_2 = -E\delta\frac{t_1+t_2}{2}A_3(\xi)\,.
\end{aligned}
\tag{8.4}
$$

Normal stresses in the edge points ($x = 0$, $z = \pm\frac{1}{2}\delta$) will be

$$
\begin{aligned}
&\sigma_1 = \pm\frac{E\beta}{2(1-\mu)}\left[2\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\frac{t_1+t_2}{2}-(t_1-t_2)\right]\\
&\sigma_2 = \pm\frac{E\beta}{2(1-\mu)}\left[2\mu\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\frac{t_1+t_2}{2}-(t_1-t_2)\mp 2(1-\mu)\frac{t_1+t_2}{2}\right].
\end{aligned}
\tag{8.5}
$$

In the outer edge points of the cylinder ($\mu = 0.3$) the stresses equal

$$
\begin{aligned}
&\sigma_1 = -E\beta(0.193t_1+1.62t_2)\\
&\sigma_2 = -E\beta(0.0580t_1+1.48t_2)
\end{aligned}
$$

in the inner

$$
\begin{aligned}
&\sigma_1 = E\beta(0.193t_1+1.62t_2)\\
&\sigma_2 = -E\beta(0.942t_1-0.486t_2)\,.
\end{aligned}
$$

At a sufficient distance from the edge

$$
\begin{aligned}
&w = -R\beta\frac{t_1+t_2}{2}\\
&\sigma_1 = \sigma_2 = \mp\frac{E\beta(t_1-t_2)}{2(1-\mu)}\,.
\end{aligned}
\tag{8.6}
$$

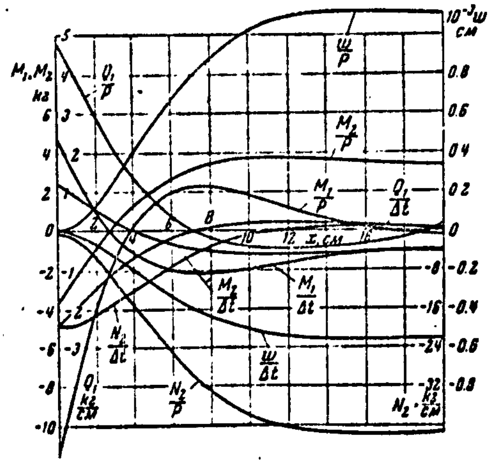

Fig. 3

Lets examine a numerical example. Let the cylindrical shell be under a uniform external pressure p and be uniformly heated to Δt^{o}C, along the thickness of the wall. We consider the heating along the axis as uniform, and the axial displacement as free. Given are

$$E_2 = 2E_1 = 2 \cdot 10^6 \text{ кг/см}^2, \qquad \mu_1 = \mu_2 \cdot 1.1 = 0.33$$
$$\beta_1 = \frac{4}{3}\beta_2 = 160 \cdot 10^{-7}\ 1/^{\circ}\mathrm{C}, \qquad \delta_1 = \delta_2 = 0.5 \text{ см}, \qquad R = 40 \text{ см}.$$

By formulas (1.16), (1.28), (2.3), (3.5), (7.2) we find

$$\begin{aligned}
B_1 &= 1.66 \cdot 10^6 \text{ кгсм}^{-1}\\
B_2 &= 0.51483 \cdot 10^6 \text{ кгсм}^{-1}\\
C_1 &= -0.13445 \cdot 10^6 \text{ кг}\\
C_2 &= -0.036126 \cdot 10^6 \text{ кг}\\
D_1 &= 0.13833 \cdot 10^6 \text{ кгсм}\\
D_2 &= 0.042902 \cdot 10^6 \text{ кгсм}\\
a &= 0.0010924 \text{ см}^{-2}\\
b &= 0.085777 \text{ см}^{-2}\\
\alpha &= 0.20577 \text{ см}^{-1}\\
\beta &= 0.20841 \text{ см}^{-1}\\
Б_1 &= 26.443 \cdot 10^{15} \text{ кг}^2\text{см}\\
Б_2 &= -8.3911 \cdot 10^{15} \text{ кг}^2\text{см}
\end{aligned}$$

$$
\begin{aligned}
Б_3 &= 26.441 \cdot 10^{15}\ \text{кг}^3 \\
Б_4 &= 9.4508 \cdot 10^{15}\ \text{кг}^3 \\
Б_5 &= -317.33 \cdot 10^{15}\ \text{кг}^3\text{см}^{-1} \\
Б_6 &= 100.69 \cdot 10^{15}\ \text{кг}^3\text{см}^{-1} \\
\Delta &= 36.362 \cdot 10^{31}\ \text{кг}^4 \\
k_1 &= 10930\ \text{кгсм}^{-1} \\
k_2 &= -3124.9\ \text{кгсм}^{-2} \\
k_3 &= 3428.7\ \text{кгсм}^{-2} \\
k_4 &= 37502\ \text{кгсм}^{-2} \\
k_5 &= 477.91\ \text{кгсм}^{-3}
\end{aligned}
$$

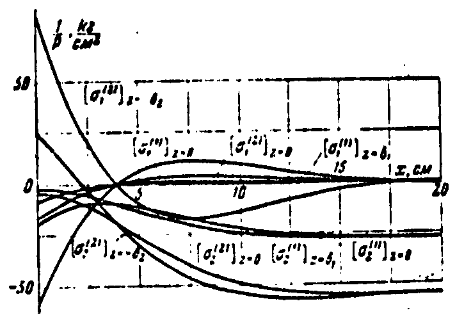

Fig. 4

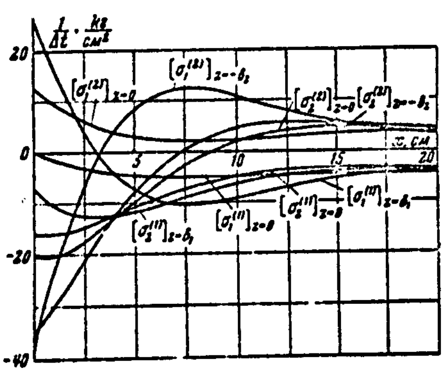

Fig. 5

Further, from formulas (1.16), (8.1) we have

$$
\begin{aligned}
f &= 29.082\Delta t \ \text{kg/cm} \\
g &= -1.3007\Delta t \ \text{kg} \\
w_0 &= -0.00053490\Delta t + \\
&+ 0.0010665p\ \text{см}
\end{aligned}
$$

Forces are calculated by formulas (8.3) which take the form of

$$M_1 = (11.66p - 5.847\Delta t)A_1(\alpha x, \beta x) - (11.51p - 5.773\Delta t)A_2(\alpha x, \beta x) - \\ - 0.1486p - 0.9802\Delta t$$

$$M_2 = (0.3663p - 0.1837\Delta t)A_1(\alpha x, \beta x) - (6.943p - 3.482\Delta t)A_2(\alpha x, \beta x) + \\ + 3.286\,p - 0.9802\Delta t$$

$$N_2 = (40p - 20.06\Delta t)A_1(\alpha x, \beta x) + (39.49p - 19.81\Delta t)A_2(\alpha x, \beta x) - 40p$$

$$w = 0.001067[1 - 0.9873A_1(\alpha x, \beta x) - A_2(\alpha x, \beta x)]p - \\ - 0.0005349[1 - 0.9873A_1(\alpha x, \beta x) - A_2(\alpha x, \beta x)]\Delta t.$$

Stresses are determined by formulas (1.11). Diagrams of forces and bendings are shown in Fig. 3, stresses of pressure in Fig. 4, and temperature in Fig.5.

b. <u>A Long Cylindrical Shell with a Supported Edge during Uniform Heating along the Meridian and with Uniform Pressure.</u>

From conditions w = 0 and M = 0 with x = 0 we find:

$$T_1 = \frac{1}{k_1}\left[\frac{C_1B_1 - C_1B_2}{B_1R}w_0 - \frac{C_1}{B_1}(N_1 + f) + g\right], \qquad T_2 = -w_0. \quad (8.7)$$

We will determine the forces by formulas (4.2) - (4.5). Let's pause in the case of a homogeneous shell with linear heating along the thickness of the wall. Here

$$T_1 = \frac{1+\mu}{2k^2}\beta\frac{t_1 - t_2}{\delta}, \qquad T_2 = R\beta\frac{t_1 + t_2}{2}. \qquad (8.8)$$

The forces and bendings are determined by formulas (5.3). On the edge of the cylinder we have

$$\sigma_1 = 0, \qquad \sigma_2 = -\tfrac{1}{2}E\beta[t_1 + t_2 \mp (t_1 - t_2)].$$

In the inner points of the fibers at the edge, the stresses would be:

$$\sigma_1 = 0, \qquad \sigma_2 = -E\beta t_1$$

in the outer

$$\sigma_1 = 0, \qquad \sigma_2 = -E\beta t_2 .$$

c. <u>A Long Cylindrical Shell with a Free Edge during Uniform Heating along the Meridian and with Uniform Pressure.</u>

Since $M_1 = Q_1 = 0$ with $x = 0$, then

$$T_1 = \frac{1}{k_1}\left[\frac{C_2B_1 - C_1B_2}{B_1R} w_0 - \frac{C_1}{B_1}(N_1 + f) + g\right], \qquad T_2 = \frac{\alpha}{\beta} T_1 \qquad (8.9) .$$

For a homogeneous cylinder with linear heating along the thickness of the wall ($p = 0$) we find /5/

$$M_1 = -\frac{E\delta^2\beta(t_1 - t_2)}{12(1-\mu)}[1 - A_3(\xi)], \qquad M_2 = -\frac{E\delta^2\beta(t_1 - t_2)}{12(1-\mu)}[1 - \mu A_3(\xi)]$$

$$Q_1 = -2(1+\mu)D\lambda\beta\frac{t_1 - t_2}{\delta}A_1(\xi), \qquad N_2 = E\delta\beta\frac{1}{2}\frac{1+\mu}{\sqrt{3(1-\mu^2)}}(t_1 - t_2)A_4(\xi)$$

$$w = -R\beta\left[\frac{t_1 + t_2}{2} + \frac{1}{2}\frac{1+\mu}{\sqrt{3(1-\mu^2)}}\beta(t_1 - t_2)A_4(\xi)\right]. \qquad (8.10)$$

Maximum stresses at the edge

$$\sigma_1 = 0, \quad \sigma_2 = \frac{E\beta(t_1 - t_2)}{2}\left[\frac{\sqrt{3(1-\mu^2)}}{3(1-\mu^2)} \mp 1\right] = \left.\begin{matrix}-0.107\\ 0.893\end{matrix}\right\} E\beta(t_1 - t_2) . \quad (\text{при } \mu = 0.3) \qquad (8.11)$$

With $kx = \pi$ the meridianal stress has an extreme value

$$\sigma_1 = \mp 1.043\frac{E\beta(t_1 - t_2)}{2(1-\mu)} = \mp 0.745E\beta(t_1 - t_2) \quad (\text{at } \mu = 0.3); \qquad (8.12)$$

there is also

$$\sigma_2 = \left[-0.0144\sqrt{3(1-\mu^2)} \mp (1 + 0.043\mu)\right]\frac{E\beta(t_1 - t_2)}{2(1-\mu)} = \left.\begin{matrix}-0.740\\ +0.706\end{matrix}\right\} E\beta(t_1 - t_2). \quad (\text{at } \mu = 0.3) \qquad (8.13)$$

d. <u>A Long Cylindrical Shell with Closely Placed Ribs during Uniform Heating along the Meridian and with Uniform Pressure.</u> (Fig. 6)

We assume that the radius of the seam surface and the radius of the central axis of the ring $R_{ш}$ differ unessentially from each other, i.e., $R_{ш} \approx R$.

We will substitute the action of the rib on the cylinder with concentrated force P kg/cm uniformly distributed along the circumference (it may be considered, for example, that a uniform pressure is distributed from the rib to the cylinder along the whole area of contact)(Fig. 6). The unknowns T_1, T_2, P are determined from conjugation conditions at the contact point of the rib and the cylinder (Fig. 6)

$$w' = 0, \qquad w = -\Delta R_{ш} = -\frac{PR^2}{E_{ш}F_{ш}} - \beta_{ш} t_{ш} R, \qquad Q_1 = -\frac{1}{2} P$$

where $\Delta R_{ш}$ is the increment of the central axis radius of the rib, $E_{ш} F_{ш}$ is the elasticity of the rib, $\beta_{ш} t_{ш}$ is the relative thermal expansion of the center axis of the rib. Then

$$T_1 = \frac{\alpha}{\beta} T_2, \qquad P = 4\alpha \frac{D_1 B_1 - C_1^2}{B_1} (\alpha^2 + \beta^2) T_2$$
$$T_2 = -\frac{w_0 + \beta_{ш} t_{ш} R}{1 + 4\frac{\alpha R^2}{E_{ш}F_{ш}} \frac{D_1 B_1 - C_1^2}{B_1} (\alpha^2 + \beta^2)} . \qquad (8.14)$$

Bending of the cylinder equals

$$w = w_0 + \frac{1}{\beta} [\alpha A_1 (\alpha x, \beta x) + \beta A_2 (\alpha x, \beta x)] T_2 . \qquad (8.15)$$

The stress in the rib equals

$$\sigma_{ш} = \frac{PR}{F_{ш}} . \qquad (8.16)$$

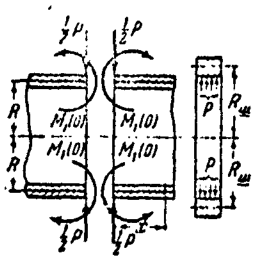

Fig. 6

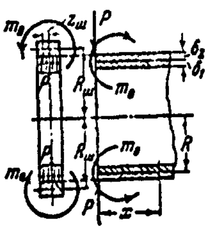

Fig. 7

With $E_ш F_ш \to \infty$ and $t_ш = 0$, we get the previously investigated instance of a shell with a rigidly fastened edge.

e. A Long Cylindrical Shell with an End Rib during Uniform Heating along the Meridian and with Uniform Pressure (Fig. 7).

The unknowns which are the constants of integration T_1 and T_2, the radial force P, and the bending moment m_0 acting on the rib are determined from equations (see /6/) (Fig. 7).

$$w = -\Delta R_ш = -\frac{PR^2}{E_ш F_ш} - \beta_ш t_ш R$$
$$w' = -\frac{m_0 R^2}{E_ш I_ш}, \quad Q_1 = -P, \quad M_1 = m_0 \qquad \text{при } x = 0 .$$

Here $E_ш I_ш$ is the bending rigidity of the rib. As a result we have

$$T_2 + w_0 = -\frac{PR^2}{E_ш F_ш} - \beta_ш t_ш R, \qquad T_1\beta - T_2\alpha = -\frac{m_0 R^2}{E_ш I_ш} \tag{8.17}$$
$$-\frac{1}{R}\frac{C_2 B_1 - C_1 B_2}{B_1}(\beta T_1 - \alpha T_2) + \frac{D_1 B_1 - C_1^2}{B_1}[\alpha(\alpha^2 - 3\beta^2)T_2 + \beta(\beta^2 - 3\alpha^2)T_1] = -P$$
$$-\frac{1}{R}\frac{C_2 B_1 - C_1 B_2}{B_1}(T_2 + w_0) + \frac{D_1 B_1 - C_1^2}{B_1}[2\alpha\beta T_1 - (\alpha^2 - \beta^2)T_2] +$$
$$+\frac{C_1}{B_1}(N_1 + f) - g = m_0 .$$

After determination of the constants it is not difficult to find the forces and stresses in the cylinder and the rib. The normal stress in the rib (Fig. 7)

$$\sigma = \frac{PR}{F_{ш}}\left(1 - \frac{m_0}{P}\frac{F_{ш}}{I_{ш}} z_{ш}\right). \tag{8.18}$$

where $z_{ш}$ is the distance from the center axis of the cross-section of the rib to the point under examination.

9. Calculation of Cylindrical Shells of Finite Length.

a. A Cylindrical Shell with Rigidly Fastened Edges during Uniform Heating along the Meridian and with Uniform Pressure.

Since $w = w' = 0$ with $x = 0$ and $x = l$, then

$$T_1 = -w_0, \quad T_2 = -w_0\frac{(\alpha^2+\beta^2)\,\mathrm{sh}\,\alpha l \sin\beta l - (\mathrm{ch}^2\alpha l - \cos^2\beta l)\,\alpha\beta}{\beta^2\mathrm{sh}^2\alpha l - \alpha^2\sin^2\beta l}$$

$$T_3 = w_0\beta\frac{(\mathrm{ch}\,\alpha l - \cos\beta l)(\beta\,\mathrm{sh}\,\alpha l - \alpha\sin\beta l)}{\beta^2\mathrm{sh}^2\alpha l - \alpha^2\sin^2\beta l} \qquad T_4 = -\frac{\alpha}{\beta}T_3. \tag{9.1}$$

For a homogeneous cylinder with temperature variations according to the linear law, along the meridian ($p = 0$) we have

$$w = -R\beta\frac{t_1+t_2}{2}\left\{1 - \frac{\mathrm{sh}\,kx\cos k(l-x) + \mathrm{ch}\,kx\sin k(l-x) + \cos kx\,\mathrm{sh}\,k(l-x)}{\mathrm{sh}\,kl + \sin kl} + \frac{\sin kx\,\mathrm{ch}\,k(l-x)}{\mathrm{sh}\,kl + \sin kl}\right\}$$

$$\vartheta = 2kR\beta\frac{t_1+t_2}{2}\frac{\mathrm{sh}\,kx\sin k(l-x) - \sin kx\,\mathrm{sh}\,k(l-x)}{\mathrm{sh}\,kl + \sin kl} \tag{9.2}$$

$$M_1 = \frac{E\delta^2\beta}{12(1-\mu)}\left\{-2\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\frac{t_1+t_2}{2(\mathrm{sh}\,kl + \sin kl)}[\mathrm{ch}\,kx\sin k(l-x) - \mathrm{sh}\,kx\cos k(l-x) - \cos kx\,\mathrm{sh}\,k(l-x) + \sin kx\,\mathrm{ch}\,k(l-x)] - (t_1 - t_2)\right\}$$

$$M_2 = \frac{E\delta\beta^2}{12(1-\mu)}\left\{-2\mu\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\frac{t_1+t_2}{2(\mathrm{sh}\,kl + \sin kl)}[\mathrm{ch}\,kx\sin k(l-x) - \mathrm{sh}\,kx\cos k(l-x) - \cos kx\,\mathrm{sh}\,k(l-x) + \sin kx\,\mathrm{ch}\,k(l-x)] - (t_1 - t_2)\right\}$$

$$Q_1 = -4Dk^3R\beta\frac{t_1+t_2}{2}\frac{\cos kx\,\mathrm{ch}\,k(l-x) - \mathrm{ch}\,kx\cos k(l-x)}{\mathrm{sh}\,kl + \sin kl}, \qquad N_1 = 0$$

$$N_2 = -E\delta\beta\frac{t_1+t_2}{2}\frac{\mathrm{sh}\,kx\cos k(l-x) + \mathrm{ch}\,kx\sin k(l-x) + \cos kx\,\mathrm{sh}\,k(l-x) + \sin kx\,\mathrm{ch}\,k(l-x)}{\mathrm{sh}\,kl + \sin kl}.$$

We calculate the stresses in the fiber points of the cylinder

(1) $x = 0$ and $x = l$

$$\sigma_1 = \pm \frac{E\delta}{2(1-\mu)}\left[2\,\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\,\frac{t_1+t_2}{2}\,\psi_2 - (t_1 - t_2)\right]$$

$$\sigma_2 = \pm \frac{E\delta}{2(1-\mu)}\left[2\mu\,\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\,\frac{t_1+t_2}{2}\,\psi_2 - (t_1 - t_2) \mp 2(1-\mu)\frac{t_1+t_2}{2}\right] \qquad (9.3)$$

(2) $x = \frac{1}{2}l$

$$\sigma_1 = \pm \frac{E\delta}{2(1-\mu)}\left[4\,\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\,\frac{t_1+t_2}{2}\,\psi_3 - (t_1 - t_2)\right]$$

$$\sigma_2 = \pm \frac{E\delta}{2(1-\mu)}\left[4\mu\,\frac{\sqrt{3(1-\mu^2)}}{1+\mu}\,\frac{t_1+t_2}{2}\,\psi_3 - (t_1 - t_2) \mp 4(1-\mu)\,\psi_4\,\frac{t_1+t_2}{2}\right] \qquad (9.4)$$

Here by means of ψ_1, ψ_2, ψ_3 and ψ_4, the following expressions are indicated:

$$\psi_1 = \frac{\operatorname{ch} kl - \cos kl}{\operatorname{sh} kl + \sin kl}$$

$$\psi_2 = \frac{\operatorname{sh} kl - \sin kl}{\operatorname{sh} kl + \sin kl} \qquad (9.5)$$

$$\psi_3 = \frac{\operatorname{ch}\frac{1}{2}kl \sin\frac{1}{2}kl - \operatorname{sh}\frac{1}{2}kl \cos\frac{1}{2}kl}{\operatorname{sh} kl + \sin kl}$$

$$\psi_4 = \frac{\operatorname{ch}\frac{1}{2}kl \sin\frac{1}{2}kl + \operatorname{sh}\frac{1}{2}kl \cos\frac{1}{2}kl}{\operatorname{sh} kl + \sin kl}$$

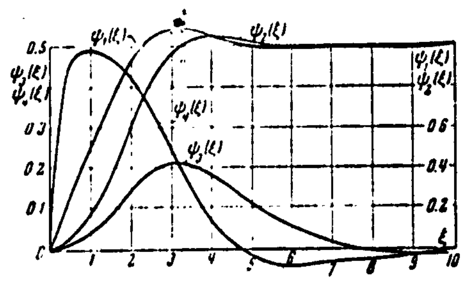

Fig. 8

The values of functions $\psi_1 = \psi_1(\xi)$, $\psi_2 = \psi_2(\xi)$, $\psi_3 = \psi_3(\xi)$, $\psi_4 = \psi_4(\xi)$ are presented in Tables VII and VIII and in Fig. 8.

Table VII

ξ	$\varphi_1(\xi)$	$\varphi_2(\xi)$	ξ	$\varphi_1(\xi)$	$\varphi_2(\xi)$
0	0	0	1.8	0.85154	0.50263
0.1	0.05000	0.00170	1.9	0.88766	0.55092
0.2	0.09999	0.00667	2.0	0.92112	0.59908
0.3	0.14999	0.014999	2.1	0.95170	0.64659
0.4	0.19997	0.026656	2.2	0.97926	0.69291
0.5	0.24991	0.041647	2.3	1.0036	0.73754
0.6	0.29978	0.059943	2.4	1.0240	0.78004
0.7	0.34953	0.081522	2.5	1.0428	0.81997
0.8	0.39908	0.10635	3.0	1.0884	0.97221
0.9	0.44836	0.13437	3.5	1.0813	1.0433
1.0	0.49724	0.16548	4.0	1.0538	1.0370
1.1	0.54557	0.19958	4.5	1.0272	1.0444
1.2	0.59320	0.23650	5.0	1.0093	1.0261
1.3	0.63991	0.27604	5.5	1.0000	1.0116
1.4	0.68550	0.31796	6.0	0.99663	1.0027
1.5	0.72972	0.36196	6.5	0.99642	0.99870
1.6	0.77231	0.40768	7.0	0.99743	0.99760
1.7	0.81301	0.45472			

Table VIII

ξ	$\varphi_3(\xi)$	$\varphi_4(\xi)$	ξ	$\varphi_3(\xi)$	$\varphi_4(\xi)$
0	0	0	2.1	0.15707	0.41250
0.1	0	0.50000	2.2	0.16733	0.39746
0.2	0.001499	0.49998	2.3	0.17692	0.38119
0.3	0.003666	0.49996	2.4	0.18571	0.36384
0.4	0.006748	0.49989	2.5	0.19357	0.34551
0.5	0.010394	0.49973	3.0	0.21615	0.24580
0.6	0.014983	0.49929	3.5	0.21085	0.14941
0.7	0.020387	0.49871	4.0	0.18580	0.072042
0.8	0.026596	0.49786	4.5	0.15170	0.017865
0.9	0.033593	0.49661	5.0	0.11628	−0.016069
1.0	0.041305	0.49482	5.5	0.083830	−0.034548
1.1	0.049801	0.49248	6.0	0.056287	−0.042182
1.2	0.058939	0.48941	6.5	0.034264	−0.042662
1.3	0.068709	0.48547	7.0	0.017629	−0.038808
1.4	0.079037	0.48059	7.5	0.0058317	−0.032702
1.5	0.089805	0.47467	8.0	−0.0018968	−0.025816
1.6	0.10091	0.46759	8.5	−0.006405	−0.019124
1.7	0.11228	0.45924	9.0	−0.008518	−0.013200
1.8	0.12372	0.44961	9.5	−0.008971	−0.0083210
1.9	0.13510	0.43858	10.0	−0.008373	−0.0045504
2.0	0.14626	0.42624			

b. <u>A Cylindrical Shell with Free Edges under the Influence of a Uniform Pressure and Temperature.</u> From the boundary conditions $M_1 = Q_1 = 0$ with $x = 0$ and $x = 1$ we determine

$$T_1 = \frac{1}{k_1}\left[\frac{C_2B_1 - C_1B_2}{B_1R} w_0 - \frac{C_1}{B_1}(N_1 + f) + g\right] \frac{(\alpha^2 + \beta^2)\,\mathrm{sh}\,\alpha l \sin\beta l - \alpha\beta(\mathrm{sh}^2\alpha l + \sin^2\beta l)}{\beta^2\mathrm{sh}^2\alpha l - \alpha^2\sin^2\beta l}$$

$$T_2 = -\frac{1}{k_1}\left[\frac{C_2B_1 - C_1B_2}{B_1R} w_0 - \frac{C_1}{B_1}(N_1 + f) + g\right], \qquad T_3 = \frac{\alpha}{\beta}T_4 \qquad (9.6)$$

$$T_4 = \frac{\beta}{k_1}\left[\frac{C_2B_1 - C_1B_2}{B_1R} w_0 - \frac{C_1}{B_1}(N_1 + f) + g\right] \frac{(\mathrm{ch}\,\alpha l - \cos\beta l)(\beta\,\mathrm{sh}\,\alpha l - \alpha\sin\beta l)}{\beta^2\mathrm{sh}^2\alpha l - \alpha^2\sin^2\beta l}.$$

For a homogeneous cylinder with linear heating along the thickness of the wall

$$w = -R\beta\frac{t_1 + t_2}{2} - \frac{1}{2}\frac{1+\mu}{k^2}\beta\frac{t_1 - t_2}{\delta}\frac{1}{\mathrm{sh}\,kl + \sin kl}[\mathrm{sh}\,kx\cos k(l - x) -$$
$$- \mathrm{ch}\,kx\sin k(l - x) - \sin kx\,\mathrm{ch}\,k(l - x) + \cos kx\,\mathrm{sh}\,k(l - x)] \quad (9.7)$$

$$\vartheta = -\frac{1+\mu}{k}\beta\frac{t_1 - t_2}{\delta}\frac{\mathrm{ch}\,kx\cos k(l - x) - \cos kx\,\mathrm{ch}\,k(l - x)}{\mathrm{sh}\,kl + \sin kl}.$$

The maximum normal stresses at the cylinder edges will be

$$\sigma_1 = 0, \qquad \sigma_2 = \frac{E\beta(t_1 - t_2)}{2}\left[\mp 1 + \frac{1+\mu}{\sqrt{3(1-\mu^2)}}\psi_2\right]. \qquad (9.8)$$

These same stresses in the middle of the cylinder, equal

$$\sigma_1 = \mp\frac{E\beta(t_1 - t_2)}{2(1-\mu)}(2\psi_4 - 1), \quad \sigma_2 = \pm\frac{E\beta(t_1 - t_2)}{2(1-\mu)}\left[2\mu\psi_4 - 1 \mp \frac{2(1-\mu^2)}{\sqrt{3(1-\mu^2)}}\psi_3\right]. \qquad (9.9)$$

c. <u>A Cylindrical Shell with Rigidly Supported Edges during Uniform Heating along the Meridian and with Uniform Pressure.</u> Since $w = M_1 = 0$ with $x = 0$ and $x = 1$, then

$$T_1 = -w_0, \quad T_2 = -\frac{1}{k_1}\left[\frac{C_2B_1 - C_1B_2}{B_1R} w_0 - \frac{C_1}{B_1}(N_1 + f) + g\right] \qquad (9.10)$$

$$T_3 = \left\{\frac{\sin^2\beta l}{k_1}\left[\frac{C_2B_1 - C_1B_2}{B_1R} w_0 - \frac{C_1}{B_1}(N_1 + f) + g\right] + w_0 \operatorname{sh}\alpha l\right\}\frac{\operatorname{ch}\alpha l - \cos\beta l}{\operatorname{sh}^2\alpha l + \sin^2\beta l}$$

$$T_4 = \left\{\frac{\operatorname{sh}\alpha l}{k_1}\left[\frac{C_2B_1 - C_1B_2}{B_1R} w_0 - \frac{C_1}{B_1}(N_1 + f) + g\right] - w_0 \sin\beta l\right\}\frac{\operatorname{ch}\alpha l - \cos\beta l}{\operatorname{sh}^2\alpha l + \sin^2\beta l}\,.$$

For a homogeneous cylinder with linear heating along the thickness we get

$$T_1 = R\beta\frac{t_1 + t_2}{2}, \qquad T_3 = -\frac{1+\mu}{k^2}\beta\frac{t_1 - t_2}{\delta} \qquad (9.11)$$

$$T_2 = -R\beta\frac{t_1 + t_2}{2}\,\frac{(\operatorname{sh} kl - \sin kl)(\operatorname{ch} kl - \cos kl)}{\operatorname{sh}^2 kl + \sin^2 kl} + 2\frac{1+\mu}{k^2}\beta\frac{t_1 - t_2}{\delta}\,\frac{[1 - F_1(kl)]\,F_4(kl)}{\operatorname{sh}^2 kl + \sin^2 kl}$$

$$T_4 = 8R\beta\frac{t_1 + t_2}{2}\,\frac{F_4(kl) + \frac{1}{4}(\operatorname{sh} 2kl - \sin 2kl)}{\operatorname{sh}^2 kl + \sin^2 kl} - 2\frac{1+\mu}{k^2}\beta\frac{t_1 - t_2}{\delta}\,\frac{[1 - F_1(kl)]\,F_2(kl)}{\operatorname{sh}^2 kl + \sin^2 kl}\,.$$

d. A Cylindrical Shell with Ribs during Uniform Heating along the Axis and with Uniform Pressure. Let's examine a cylindrical shell with close and evenly placed ribs. In this instance, the ribs at a distance from the edges act only through stretch on the compression. The ribs transmit only the concentrated ringed force P kg/cm. to the shell. We have five boundary conditions for determining the five unknowns, T_1, T_2, T_3, T_4, P. We initially select the middle coordinate between the two ribs (assuming $R_{ш} \approx R$). Then

$$w' = 0,\ Q_1 = 0 \ \text{при}\ x = 0; \quad w' = 0,\ \Delta R_{ш} = -w,\ Q_1 = \tfrac{1}{2}P \ \text{at}\ x = \tfrac{1}{2}l\,.$$

Conditions with x = 0 follow from symmetric consideration of deformation relative to the initial section. As a result

$$T_1 = \frac{\alpha\Phi_4\left(\frac{1}{2}l\right) + \beta\Phi_3\left(\frac{1}{2}l\right)}{\beta\Phi_4\left(\frac{1}{2}l\right) - \alpha\Phi_3\left(\frac{1}{2}l\right)} T_2 \tag{9.12}$$

$$P = 4\alpha\beta\left(\alpha^2 + \beta^2\right) \frac{\Phi_4^2\left(\frac{1}{2}l\right) + \Phi_3^2\left(\frac{1}{2}l\right)}{\beta\Phi_4\left(\frac{1}{2}l\right) - \alpha\Phi_3\left(\frac{1}{2}l\right)} \frac{D_1B_1 - C_1^2}{B_1} T_2$$

$$T_2 = -\left(w_0 + \beta_{\text{ш}} t_{\text{ш}} R\right)\left[\beta\Phi_4\left(\tfrac{1}{2}l\right) - \alpha\Phi_3\left(\tfrac{1}{2}l\right)\right] : \left\{\Phi_1\left(\tfrac{1}{2}l\right)\left[\alpha\Phi_4\left(\tfrac{1}{2}l\right) + \right.\right.$$
$$\left. + \beta\Phi_3\left(\tfrac{1}{2}l\right)\right] + \Phi_2\left(\tfrac{1}{2}l\right)\left[\beta\Phi_4\left(\tfrac{1}{2}l\right) - \alpha\Phi_3\left(\tfrac{1}{2}l\right)\right] + 4\alpha\beta\left(\alpha^2+\beta^2\right)\left[\Phi_4^2\left(\tfrac{1}{2}l\right) + \right.$$
$$\left.\left. + \Phi_3^2\left(\tfrac{1}{2}l\right)\right] \frac{D_1B_1 - C_1^2}{B_1 E_{\text{ш}} F_{\text{ш}}} R^2\right\}, \qquad T_3 = T_4 = 0 \cdot$$

From here with $E_{\text{ш}}F_{\text{ш}} \to \infty$ and $t_{\text{ш}} = 0$, we get the instance of rigid fastening of the cylinder during $x = \mp \frac{1}{2} l$.

10. Static Stability. We assume that before the loss of stability, the shell is uniformly compressed by forces $N_1 = -q$ kg/cm. Let's examine the instance of an axially symmetric form of stability when the following conditions are satisfied:

$$w(0) = w(l) = w_0, \qquad w''(0) = w''(l) = 0 \cdot$$

Solution of basic equation

$$w^{IV} + 2aw'' + b^2\left(w - w_0\right) = 0 \tag{10.1}$$

where

$$w_0 = -\frac{B_1}{Rb^2} \frac{q}{D_1B_1 - C_1^2} \tag{10.2}$$

we search in form of

$$w = w_0 + w_\theta \cdot \tag{10.3}$$

Here w_θ is the bending due to the loss of stability. Let

$$w_\theta = \sum_{\theta=1}^{\theta=\infty} A_\theta \sin\frac{\lambda x}{R} ; \tag{10.4}$$

by this $\lambda = \theta\pi R/l$, θ is the number of half-waves in the direction of the shell axis originating during the loss of stability. By substituting expressions (10.3) and (10.4) into equation (10.1) we find the critical force

$$q^* = \frac{1}{B_1}\left[\frac{\lambda^2}{R^2}(D_1B_1 - C^2_1) + \frac{B_1^2 - B_2^2}{\lambda^2}\right] - 2\frac{C_2B_1 - C_1B_2}{B_1R} . \qquad (10.5)$$

By considering q^* as a continuous function λ we determine the minimum value of the critical force

$$q_{\min}{}^* = \frac{2}{B_1R}\left[\sqrt{(B_1^2 - B_2^2)(D_1B_1 - C_1^2)} - C_2B_1 + C_1B_2\right] \qquad (10.6)$$

which occurs during

$$\lambda = \sqrt[4]{\frac{B_1^2 - B_2^2}{D_1B_1 - C_1^2}R^2} . \qquad (10.7)$$

From (10.6), (10.7) we obtain the known form given by S. P. Timoshenko for a homogeneous shell /7/

$$q_{\min}{}^* = \frac{E\delta^2}{R}\frac{1}{\sqrt{3(1-\mu^2)}} , \qquad \lambda = \sqrt[4]{12(1-\mu^2)}\sqrt{\frac{R}{\delta}} . \quad (10.8)$$

Stresses in the layers at the moment of stability loss are determined from equations

$$\sigma_1^{(1)}\delta_1 + \sigma_1^{(2)}\delta_2 = -q^*_{\min}, \qquad \frac{\sigma_1^{(1)}}{E_1} = \frac{\sigma_1^{(2)}}{E_2} ;$$

and thus:

$$\sigma_1^{(1)} = -q^*_{\min}\frac{E_1}{E_1\delta_1 + E_2\delta_2} , \qquad \sigma_1^{(2)} = -q^*_{\min}\frac{E_2}{E_1\delta_1 + E_2\delta_2} . \quad (10.9)$$

From here /7/ we obtain the following for a homogeneous shell:

$$\sigma_1 = -\frac{q_{\min}{}^*}{\delta} = -\frac{E\delta}{R}\frac{1}{\sqrt{3(1-\mu^2)}} . \qquad (10.10)$$

The numerical values of forces q min,* stresses $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ for a series of values σ_1, σ_2, E_1, E_2, μ_1, μ_2 are given in Table IX. In Tables X - XI the values of stresses in the layer during the loss of stability are shown for a series of relationships δ_1/δ_2 and E_1/E_2 for value $\mu = 0.3$.

These results are presented in Fig. 10 - 11, where

$$\eta_1^{(1)} = -\frac{\sigma_1^{(1)} R}{E_1 \delta}$$
$$\eta_1^{(2)} = -\frac{\sigma_1^{(2)} R}{E_1 \delta}. \qquad (10.11)$$

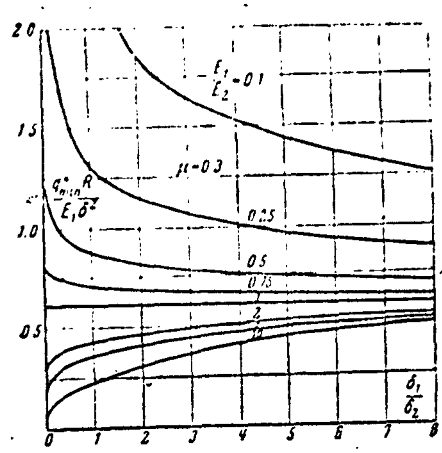

Fig. 9.

11. Dynamic Stability. The problem concerning dynamic stability of a bar was raised by N.M. Belyaev /8/.

Let's investigate an axially symmetric form of loss of stability of a bimetallic shell when the latter is uniformly compressed by axial forces

$$N_1 = -q = -q_0 - q_1 \cos \omega T \qquad (11.1)$$

where w is the circular frequency of the external force and T is the time. Considering the inertia of the bending of the shell sections, the equation of the problem will appear as

$$w^{IV} + 2aw'' + b^2 w = \frac{B_1}{D_1 B_1 - C_1^2}\left(\frac{1}{R}\frac{B_2}{B_1}N_1 - \Gamma\frac{d^2 w}{dT^2}\right) \qquad (11.2)$$

where

$$\Gamma = \gamma_1\delta_1 + \gamma_2\delta_2 \qquad (11.3)$$

where γ_1 and γ_2 are specific gravities of the material layers.

Solution of equation (11.2) is presented as

$$w = w_0 + \sum_{\theta=1}^{\theta=\infty} A_0(T)\sin\frac{\lambda x}{R} \qquad (11.4)$$

where $A_0(T)$ is the parameter, depending on time,

$$w_0 = -\frac{B_2 R}{B_1^2 - B_2^2}(q_0 + q_1\cos\omega T). \qquad (11.5)$$

By substituting expression (11.4) into equation (11.2) we find

$$\frac{\Gamma B_1}{D_1 B_1 - C_1^2}\sum_{\theta=1}^{\theta\to\infty}\left\{\frac{d^2 A_0(T)}{dT^2} + \frac{\lambda^2}{R^2}(q^* - q_0 - q_1\cos\omega T)A_0(T)\right\}\sin\frac{\lambda x}{R} = 0 \qquad (11.6).$$

Here q^* is the critical force determined by formula (10.5). Equation (11.6) is reduced to the Mathieu equation.

$$\frac{d^2 A_0(T)}{d\tau^2} + 4\frac{\omega_0^2}{\omega^2}(1-\varepsilon_0\cos 2\tau)A_0(T) = 0 \qquad (11.7)$$

where w_o is the average natural frequency of the shell,

$$\tau = \frac{1}{2}\omega T, \qquad \varepsilon_0 = \frac{q_1}{q^* - q_0}, \qquad \omega_0 = \frac{\lambda}{R}\sqrt{\frac{q^* - q_0}{\Gamma_0}}. \qquad (11.8)$$

Generally speaking, investigation of dynamic stability of an elastic system always reduces to the Mathieu equation if the forms of loss of static stability and oscillations are the same.

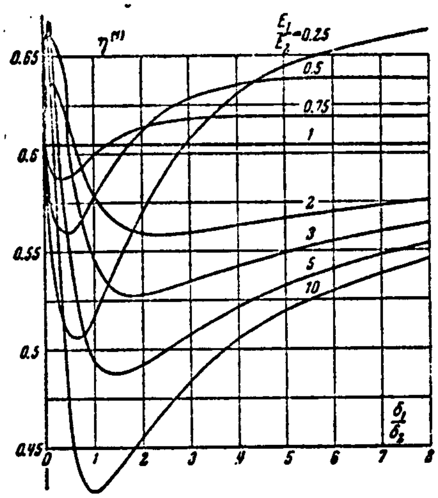

Fig. 10

Mathieu equation was investigated more comprehensively for the first time by A. Andronov and M. Leontovich /9/ in connection with the problem of oscillation of a pendulum in the gravitational field. For instance, when $\epsilon_o < 1$ and w_e is any valid number, they showed that equation (11.7) can have the following solution. The first, a so-called stable solution,

represents an almost periodic function

$$A_\theta(T) = g_1 e^{i\tau T}\varphi(\tau) + g_2 e^{-i\tau T}\varphi(-\tau) = g_1' \cos \tau T \eta(\tau) + g_2' \sin \tau \nu(\tau) \tag{11.9}$$

where $\varphi(\tau)$, $\varphi(-\tau)$, $\eta(\tau)$, $\nu(\tau)$ are functions of τ, the period of which equals π; τ is the characteristic number; g_1, g_2, g_1', g_2' are the arbitrary constants.

If τ is the rational number, then $A_\theta(T)$ is a function with a period 2π.

The stable values w_θ/w correspond to the stable solution (11.9). The second to the unstable solution (when $\tau = ik$ or $\tau = 1 + ik$)

$$A_\theta(T) = g_1 e^{-\tau k}\varphi_1(\tau) + g_2 e^{\tau k}\varphi_1(-\tau), \tag{11.10}$$

where $\varphi_1(\tau)$ is the periodic function with a period π or 2π. Here $A_\theta(T)$ increases indefinitely only if the correlation between $A_\theta(0)$ and $dA_\theta(0)/d\tau$ are not such that $g_2 = 0$.

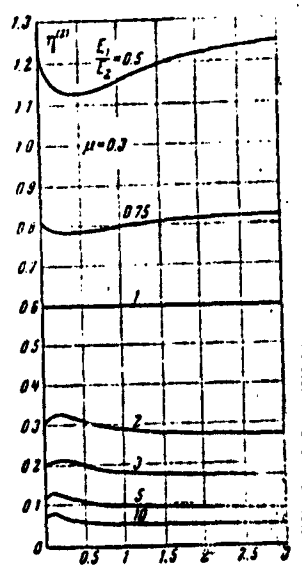

Fig. 11

Table IX

δ_1, см	δ_2, см	$10^{-6}E_1$, кг/см^2	$10^{-6}E_2$, кг/см^2	μ_1	μ_2	$10^{-4}q^*_{min}R$, кг	$10^{-4}\sigma_1^{(1)}R$, кг/см	$10^{-4}\sigma_1^{(2)}R$, кг/см
0.05	0.05	2	1	0.3	0.33	0.8857	0.590	1.18
0.05	0.05	1	2	0.33	0.3	0.8634	1.15	0.576
0.05	0.05	2	2	0.3	0.3	1.210	1.21	1.21
0.05	0.05	1	1	0.33	0.33	0.6116	0.612	0.612
0.07	0.05	2	1	0.3	0.33	1.315	0.692	1.38
0.07	0.05	1	2	0.33	0.3	1.206	1.42	0.710
0.05	0.07	2	1	0.3	0.33	1.239	0.729	1.46
0.05	0.07	1	2	0.33	0.3	1.286	1.35	0.677
0.15	0.05	2	1	0.3	0.33	3.970	1.13	2.27
0.15	0.05	1	2	0.33	0.3	3.128	2.50	1.25
0.05	0.15	2	1	0.3	0.33	3.208	1.28	2.56
0.05	0.15	1	2	0.33	0.3	3.912	2.24	1.12

Table X

Values $\eta^{(1)}$ With $\mu = 0.3$

E_1 / E_2	10	8	5	3	2	1	0.75	0.5	0.25	0.1
δ_1 / δ_2										
∞	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052
10	0.5573	0.5580	0.5639	0.5721	0.5816	0.6052	0.6177	0.6367	0.6643	0.6539
8	0.5471	0.5492	0.5555	0.5659	0.5775	0.6052	0.6191	0.6389	0.6623	0.6344
5	0.5203	0.5240	0.5344	0.5509	0.5683	0.6052	0.6210	0.6394	0.6459	0.5795
3	0.4842	0.4908	0.5089	0.5354	0.5607	0.6052	0.6195	0.6297	0.6087	0.5101
2	0.4548	0.4652	0.4925	0.5291	0.5605	0.6052	0.6146	0.6136	0.5706	0.4622
1	0.4271	0.4473	0.4942	0.5455	0.5795	0.6052	0.6006	0.5795	0.5171	0.4271
0.75	0.4330	0.4575	0.5108	0.5634	0.5932	0.6052	0.5949	0.5686	0.5074	0.4327
0.5	0.4622	0.4909	0.5476	0.5944	0.6136	0.6052	0.5895	0.5605	0.5077	0.4548
0.25	0.5493	0.5763	0.6194	0.6398	0.6367	0.6052	0.5881	0.5644	0.5314	0.5052
0.1	0.6539	0.6640	0.6686	0.6547	0.6367	0.6052	0.5942	0.5816	0.5671	0.5573
0	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052	0.6052

Table XI

Values $\eta^{(2)}$ with $\mu = 0.3$

E_1/F_0 / δ_1/δ_0	10	8	5	3	2	1	0.75	0.5	0.25	0.1
∞	0.00605	0.00945	0.0242	0.0672	0.1513	0.6052	1.076	2.421	9.684	6.052
10	0.05573	0.06987	0.1128	0.1907	0.2908	0.6052	0.8236	1.273	2.657	6.539
8	0.05471	0.06866	0.1111	0.1886	0.2887	0.6052	0.8254	1.278	2.649	6.344
5	0.05203	0.06550	0.1069	0.1836	0.2842	0.6052	0.8279	1.279	2.584	5.795
3	0.04842	0.06135	0.1018	0.1785	0.2804	0.6052	0.8200	1.259	2.435	5.101
2	0.04548	0.05815	0.09849	0.1764	0.2803	0.6052	0.8194	1.227	2.282	4.622
1	0.04271	0.05592	0.09883	0.1818	0.2897	0.6052	0.8008	1.150	2.068	4.271
0.75	0.04330	0.05719	0.1022	0.1878	0.2966	0.6052	0.7932	1.137	2.030	4.327
0.5	0.04622	0.06136	0.1095	0.1981	0.3068	0.6052	0.7850	1.121	2.031	4.548
0.25	0.05493	0.07204	0.1239	0.2133	0.3183	0.6052	0.7841	1.129	2.125	5.032
0.1	0.06529	0.08299	0.1347	0.2182	0.3183	0.6052	0.7923	1.163	2.268	5.573
0	0.06052	0.07565	0.1210	0.2017	0.3026	0.6052	0.8070	1.210	2.421	6.052

The first solution corresponds to the stable work region of the shell, the second to the unstable. Values ε_θ and w_θ/w determining the dividing lines of the stable and unstable regions correspond to the solution, one of which is the periodic with a period π or 2π, the other being $\tau F(\tau) + \Phi(\tau)$, where F (τ) and $\Phi(\tau)$ are the periodic function with a period of π or 2π.

The boundaries of the unstable regions (with accuracy to $\epsilon_\theta^{\ 3}$) will be /10/:

First region

$$\left(\frac{2\omega_0}{\omega_{кр}}\right)_{11,12} = \sqrt{1 \mp \tfrac{1}{2}\varepsilon_0 + \tfrac{1}{32}\varepsilon_0^2 \mp \tfrac{55}{512}\varepsilon_0^3} \qquad (11.11)$$

Second region

$$\left(\frac{2\omega_0}{\omega_{кр}}\right)_{21} = \sqrt{4 - \tfrac{1}{3}\varepsilon_0^2}, \qquad \left(\frac{2\omega_0}{\omega_{кр}}\right)_{22} = \sqrt{4 + \tfrac{5}{3}\varepsilon_0^2} \qquad (11.12)$$

Third region

$$\left(\frac{2\omega_0}{\omega_{кр}}\right)_{31,32} = \sqrt{9 + \tfrac{81}{64}\varepsilon_0^2 \mp \tfrac{9}{8}\varepsilon_0^3} \qquad (11.13)$$

Here w_{kp} are the critical values of frequencies of the external force q, during which the cylindrical shell becomes unstable.

12. A Multilayered Cylindrical Shell. If the shell is multilayered, then all basic correlations obtained previously remain the same, and the expression changes for B_1, B_2, ..., D_2, f and g. Let the shell have m layers. Let's assume that the surface in n layers, distant from the internal surface, is the original. The surfaces of the j layer, are spaced at a distance of ξ_j and ξ_{j-1}.

Let $\delta_j = \xi_j - \xi_{j-1}$ be the thickness of the j layer, E_j the modulus of normal elasticity, μ_j the Poissons ratio, β_j the coefficient of the linear thermal expansion. As previously, we consider that the shell is thin and elastic. Then, instead of expression (1.1) for a bimetallic shell, we have the following correlations for a multilayered shell:

$$\begin{aligned}
M_1 &= \sum_1^n \int_{\xi_{j-1}}^{\xi_j} \sigma_1^{(j)} z\,dz + \sum_{n+1}^m \int_{-\xi_j}^{-\xi_{j-1}} \sigma_1^{(j)} z\,dz \\
M_2 &= \sum_1^n \int_{\xi_{j-1}}^{\xi_j} \sigma_2^{(j)} z\,dz + \sum_{n+1}^m \int_{-\xi_j}^{-\xi_{j-1}} \sigma_2^{(j)} z\,dz \\
N_1 &= \sum_1^n \int_{\xi_{j-1}}^{\xi_j} \sigma_1^{(j)}\,dz + \sum_{n+1}^m \int_{-\xi_j}^{-\xi_{j-1}} \sigma_1^{(j)}\,dz \\
N_2 &= \sum_1^n \int_{\xi_{j-1}}^{\xi_j} \sigma_2^{(j)}\,dz + \sum_{n+1}^m \int_{-\xi_j}^{-\xi_{j-1}} \sigma_2^{(j)}\,dz
\end{aligned} \qquad (12.1)$$

Here the stresses in the j-m layer $\sigma_1^{(j)}$ and $\sigma_2^{(j)}$ are determined by Hooke's law. In order to obtain formulas for $\sigma_1^{(j)}$ and $\sigma_2^{(j)}$ it is sufficient to exchange indexes 1 and 2 related to the layers in expression (1.11) by j.

As a result, for internal forces of a multilayered shell, expressions which in their form do not coincide with formulas (1.12)- (1.15) are obtained; however, for rigidity the following expressions should be taken:

$$B_1 = \sum_1^m \frac{E_j \delta_j}{1-\mu_j^2}, \quad B_2 = \sum_1^m \mu_j \frac{E_j \delta_j}{1-\mu_j^2}, \quad \Gamma = \sum_1^m \gamma_j \delta_j \tag{12.2}$$

$$D_1 = \frac{1}{3} \sum_1^m \frac{E_j (\xi_j^3 - \xi_{j-1}^3)}{1-\mu_j^2}, \qquad D_2 = \frac{1}{3} \sum_1^m \mu_j \frac{E_j (\xi_j^3 - \xi_{j-1}^3)}{1-\mu_j^2}$$

$$C_1 = \frac{1}{2} \sum_1^n \frac{E_j \delta_j (\xi_j + \xi_{j-1})}{1-\mu_j^2} - \frac{1}{2} \sum_{n+1}^m \frac{E_j \delta_j (\xi_j + \xi_{j-1})}{1-\mu_j^2}$$

$$C_2 = \frac{1}{2} \sum_1^n \mu_j \frac{E_j \delta_j (\xi_j + \xi_{j-1})}{1-\mu_j^2} - \frac{1}{2} \sum_{n+1}^m \mu_j \frac{E_j \delta_j (\xi_j + \xi_{j-1})}{1-\mu_j^2}$$

$$f = \sum_1^n \frac{E_j \delta_j m_j}{1-\mu_j} + \sum_{n+1}^m \frac{E_j \delta_j m_j'}{1-\mu_j}, \qquad g = \frac{1}{2} \sum_1^n \frac{E_j \delta_j^2 n_j}{1-\mu_j} + \frac{1}{2} \sum_{n+1}^m \frac{E_j \delta_j^2 n_j'}{1-\mu_j}.$$

$$m_j = \frac{1}{\delta_j} \int_{\xi_{j-1}}^{\xi_j} \beta_j t \, d., \qquad m_j' = \frac{1}{\delta_j} \int_{-\xi_j}^{-\xi_{j-1}} \beta_j t \, dz$$

$$n_j = \frac{2}{\delta_j^2} \int_{\xi_{j-1}}^{\xi_j} \beta_j t z \, dz, \qquad n_j' = \frac{2}{\delta_j^2} \int_{-\xi_j}^{-\xi_{j-1}} \beta_j t z \, dz.$$

Further calculations remain unchanged.

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